代数 1 H 班 作业 14

2022 年 12 月 22 日

In the following, K is a field such that $char(K) \neq 2$, and (V, g) is a finite dimensional vector space over field K with nondegenerate bilinear form g.

题 1. *Consider isomorphism classes of finite dimensional vector spaces over K. We can define addition by taking direct sum. What is the abelian group obtained from Grothendieck construction.*

题 2. *Show the product structure on the extended square class group $Q(K) = \mathbb{Z}/2\mathbb{Z} \times K^\times/(K^\times)^2$ defines an abelian group. In other words, $a_i \in \mathbb{Z}/2\mathbb{Z}$ and $b_i \in K^\times/(K^\times)^2$, we define $(a_1, b_1) \cdot (a_2, b_2) = (a_1 + a_2, (-1)^{a_1 a_2} b_1 b_2)$. Verify this defines a group structure.*

题 3. *In class, we proved that $W(\mathbb{F}_p) \cong Q(\mathbb{F}_p)$. Please complete the last part of the structure theorem about Witt group for finite fields. Prove that the extended square class group for a finite field $Q(\mathbb{F}_p)$ is isomorphic to*

1. *If $p \equiv 1 \mod 4$, $Q(\mathbb{F}_p) \cong \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$.*
2. *If $p \equiv 3 \mod 4$, $Q(\mathbb{F}_p) \cong \mathbb{Z}/4\mathbb{Z}$.*

Find (V, g) representing the four elements $W(\mathbb{F}_p)$ and describe the ring structure induced by tensor product.

题 4. *For (V, g) over finite field $K = \mathbb{F}_p$, is every element $a \in \mathbb{F}_p$ representable by $a = g(v, v)$ for some nonzero vector $v \in V$? Please find those (V, g) having this proposition and those who doesn't.*

题 5. *If K is a subfield in F, we can view symmetric matrices with entries in K as symmetric matrices with entries in F. Show that this induces group homomorphisms $GW(K) \to GW(F)$ and $W(K) \to W(F)$. Find examples such that the homomorphisms are*

1. *isomorphisms.*
2. *not injective.*
3. *not surjective.*

题 6. *For field K, if any two-dimensional (V, g) is isotropic, prove that any element $a \in K$ has a square root $b \in K$ such that $b^2 = a$. Calculate $GW(K)$ and $W(K)$ in this case.*