代数 1 H 班 作业 2

2022 年 9 月 16 日

题 1. *Find the number of isomorphism classes of actions of* S_4 *on* $[5]$.

题 2. *Prove that there is no transitive action of* S_6 *on* $[7]$. *(How about* S_7 *and* $[8]$*?)*

题 3. *Suppose that the map* $f : G \longrightarrow G$ *by* $a \mapsto a^{-1}$ *is an automorphism of group* G*, then* G *is abelian.*

题 4. *Let* G *be a group generated by real valued functions* $f = \frac{1}{x}$ *and* $g = \frac{x-1}{x}$ *via composition of functions. Prove that* G *is isomorphic to* S_3.

题 5. *Classify subgroups and normal subgroups of* D_n *for* $n \geq 3$.

题 6. *Prove that* $G = GL(2, \mathbb{F}_2)$ *is isomorphic to* S_3 *by the action of* G *on* $(\mathbb{F}_2)^2$.

题 7. ***Second Isomorphism Theorem****. Let* H *be a normal subgroup of group* G *and* K *be a subgroup of* G*. Prove*

1. $HK = \{hk | h \in H, k \in K\}$ *is a subgroup of* G.
2. $H \cap K$ *is a normal subgroup of* K.
3. *There is an isomorphism* $HK/H \cong K/H \cap K$.

题 8. *Let* $O_1, \cdots, O_k$ *be all the conjugacy classes in a finite group* G*. Choose* $x_i \in O_i$ *and let* $C_i = \{g \in G | g x_i g^{-1} = x_i\}$ *(which is called the centralizer of* x_i*). Denote* $n_i = |C_i|$*. Prove*

$$\frac{1}{n_1} + \frac{1}{n_2} + \cdots \frac{1}{n_k} = 1$$

题 9. ***Artin, Chapter 6, 8.1*** *Does the rule* $P * A = PAP^T$ *define an operation of* $\mathrm{GL}(n, \mathbb{R})$ *on the set of real* $n \times n$ *matrices? (Here* P^T *means the transpose of* P*)*

题 10. 定义 $PGL(2, \mathbb{F}_3) = GL(2, \mathbb{F}_3)/D$. 其中 $D = \{\lambda I_5 \mid \lambda \in \mathbb{F}_5^\times\}$. 证明 $PGL(2, \mathbb{F}_3) \cong S_4$. 提示：考虑 $GL(2, \mathbb{F}_3)$ 在 $(\mathbb{F}_3)^2$ 的所有一维子空间组成的集合上的作用.

题 11. 令 G 是一个群, $\mathbb{R}^\times$ 是 $\mathbb{R}$ 中非零元素在域的乘法下组成的群。考虑由 $G \to \mathbb{R}$ 的所有映射组成的 $\mathbb{R}$-线性空间 V. 假设 S 是由有限个 $G \to \mathbb{R}^\times$ 的群同态组成的集合. 证明 S 中的元素在 V 上线性无关.

题 12. 证明有限群 G 是循环群当且仅当对任意正整数 n, G 至多只有一个阶数为 n 的子群.

题 13. *(Semidirect product) Let* H *and* K *be two groups and* $\phi\colon K \to \mathrm{Aut}(H)$ *be a group homomorphism. Define a binary operation on* $H \times K$ *by* $(h, k)(h', k') = (h\varphi(k)(h'), kk')$. *Check this binary operation gives a group structure. Prove that the subsets* $\{e_H\} \times K$ *and* $H \times \{e_K\}$ *are subgroups of this group and* $H \times \{e_K\}$ *is a normal subgroup. Find one example that* $\{e_H\} \times K$ *is not a subgroup.*

题 14. *Prove that* $\mathrm{GL}(n, \mathbb{C})$ *is isomorphic to a subgroup of* $\mathrm{GL}(2n, \mathbb{R})$.

题 15. 证明 $\mathbb{C}^\times$ 同构于 $\mathbb{C}/\mathbb{Z}$. 其中 $\mathbb{C}^\times$ 是 $\mathbb{C}$ 中非零元素组成的乘法群. $\mathbb{C}$ 和 $\mathbb{Z}$ 是加法群.

题 16 (思考题，不用交)**.** 证明 $PGL(2, \mathbb{F}_5) \cong S_5$.

题 17 (思考题，不用交，在学完模论之后有更多工具可以做)**.** *Let* G *be the group* $\mathrm{GL}(3, \mathbb{F}_2)$.

1. *How many conjugacy classes does* G *have?*
2. *Show that* G *has exactly two conjugacy classes of size 24.*