代数 1 H 班 作业 5

2022 年 10 月 19 日

题 1. ***Artin, Chapter 11, 1.7 (a)***
Let U be an arbitrary set and R be the set of subsets in U. Addition and multiplication of elements of R are defined by $A + B = A \cup B - A \cap B$ and $A \cdot B = A \cap B$. Prove that R is a ring.

题 2. *Determine whether the division with reminder $g(x) = f(x)q(x) + r(x)$ exists in $R[x]$ for the following $R, f(x), g(x)$. If it exists, find the $q(x), r(x)$.*

1. $R = \mathbb{Z}, f(x) = 2x^2 + x + 1, g(x) = 2x^3 + 7x^2 + 4x + 8$
2. $R = \mathbb{Z}/6\mathbb{Z}, f(x) = 2x + 1, g(x) = 2x^2 + 2x$
3. $R = \mathbb{Z}/6\mathbb{Z}, f(x) = 5x + 1, g(x) = 2x^2 + 2x$

题 3. ***Artin, Chapter 11, 1.8***
Determine the units in $\mathbb{Z}/12\mathbb{Z}$, $\mathbb{Z}/8\mathbb{Z}$.

题 4. *Let R be a ring and I, J ideals of R. Prove the following*

1. *$I \cap J$ is an ideal of R,*
2. *$I + J = \{a + b | a \in I, b \in J\}$ is an ideal of R,*
3. *$IJ = \{\sum_{i=0}^{n} a_i b_i | a_i \in I, b_i \in J, n \in \mathbb{Z}_{\geq 0}\}$ is an ideal of R.*

题 5. ***Artin Chapter 11, 3.4*** *Let $\phi\colon \mathbb{C}[x, y] \to \mathbb{C}[t]$ defined by $x \mapsto t + 1$ and $y \mapsto t^3 - 1$. Determine the kernel K of ϕ and prove that every ideal I of $\mathbb{C}[x, y]$ that contains K can be generated by two elements.*

题 6 (Nilpotent groups)**.**

定义 1. *Let* $G_1 = G/C(G)$, *and* $G_{n+1} = G_n/C(G_n)$. *We call a group* G *unipotent if and only if* G_m *has order* 1 *for some* m.

1. *Prove that p-groups have nontrivial center and hence nilpotent.*
2. *Prove that a finite group is nilpotent if and only if it is the product of its Sylow subgroups.*
3. *Prove that if* G *is nilpotent, then the subgroups and quotient groups of* G *are also nilpotent.*
4. *Prove that the group* U *consisting of* $n \times n$ *upper triangular matrices with elements in a ring* R *and diagonal elements being* 1 *is nilpotent.*
5. *Prove that a finite group* G *is nilpotent if and only if it is a isomorphic to a subgroup of* U *for some* n *and* R.
6. *Prove that if* G *is nilpotent and* $[G : G] = G$, *then* G *has only one element.*

题 7. *Use the notation from homework 1, question 5. Prove the Bruhat decomposition.*

$$GL(n, F) = \sqcup_{w \in W} BwB. \tag{1}$$

题 8 (Parabolic subgroup of $PSL(n, F)$)**.** *Let* $n \geq 3$ *and* F *a field. Denote by* $G = SL(n, F)$ *the group of matrices with determinant* 1.

1. *Let* B *be the subgroup of* G *consisting of upper triangular matrices,* T *the subgroup of* G *consisting of diagonal matrices,* N *the subgroup consisting of matrices with exactly one nonzero element in each row and column. Prove that* T *is a normal subgroup of* N, *when* $|F| \geq 3$, *then* N *is the normalizer of* T. *Prove that* B *and* N *generates* G.
2. *Let* e_i *be the column vector in* F^n *with* i*-th component* 1 *and other components* 0. *Prove that the multiplication of matrices with vectors induces an group operation of* $W = N/T$ *on the set* $\{Span(e_1), \cdots, Span(e_n)\}$. *Identify* $Span(e_i)$ *with* $i \in [n]$. *Prove this action induces an isomorphism* $f\colon N/T \to S_n$.

3. *Fixing $w \in S_n$, let $\tilde{w} \in N$ be a representative in $f^{-1}(w)$. Prove that $\tilde{w}B$, $B\tilde{w}$ and $B\tilde{w}B$ do not depend on the choice of $\tilde{w}$. So we can denote by BwB for $B\tilde{w}B$. Prove the following decomposition.*

$$G = \sqcup_{w \in W} BwB. \tag{2}$$

4. *Let $\{s_1, \cdots, s_{n-1}\}$ be the set of fundamental transpositions. Prove that s_i are not in the normalizer of B and $s_iBw \subset BwB \sqcup Bs_iwB$. In general, when S_n is replaced by other Coxeter groups (not necessarily finite), such a structure is called a (B, N)-pair.*

5. *Let π be a subset of fundamental transpositions $\{s_1, \cdots, s_{n-1}\}$. Let W_π be the subgroup of S_n generated by π. Prove that*

$$P_\pi = \sqcup_{w \in W_\pi} BwB$$

is a subgroup of G.

6. *Prove that any subgroup P of G containing B is of the form P_π. We call them parabolic subgroups. (Hint: if $\tilde{\omega} = \tilde{s}_{i_1} \cdots \tilde{s}_{i_l} \in P$ with length $l(w) = l$, try to prove all $\tilde{s}_{i_j} \in P$.)*

7. *Count the number of parabolic subgroups.*

题 9 (Simplicity of $PSL(n, F)$)**.** *Following the notation in last question. Let U be the subgroup of G consisting of upper triangular matrices with diagonal elements being* 1*. Prove that*

1. *Show that the center of G is*

$$Z = \{\lambda I | \lambda^n = 1\}.$$

2. *G is generated by conjugates of U.*

3. *$G = [G, G]$.*

4. *The intersection of conjugates of B is the center of G.*

5. *Let H be a normal subgroup of G, then either $H \subset Z$ or $HU = G$. (Hint: 1. Use the classification of parabolic groups. 2. Take a look at the proof in the class. 3. Prove that if $s_i \in HU$, then the nearby $s_j \in HU$.)*

6. *Let H be a normal subgroup of G, then either $H \subset Z$ or $H = G$.*

7. *Prove that $PSL(n, F)$ is simple for $n \geq 3$.*

题 10. *Let V be a n-dimensional vector space over field F.*

定义 2 (Flag)**.** *A flag F is defined to be a chain of subspaces*

$$F : \{0\} = V_0 \subsetneq V_1 \subsetneq V_2 \cdots \subsetneq V_n = V$$

Denote by FL the set of flags.

Let $G = GL(V)$, and define an action of G on the set of flags by

$$g \cdot F \colon \{0\} = V_0 \subset g(V_1) \subset g(V_2) \cdots \subset V_n = V.$$

定义 3 (Borel subgroup)**.** *Let F be a flag, the stablizer of F is denoted by B and called the Borel subgroup of G.*

Fixing a flag F and the corresponding Borel subgroup B, prove that the linear action of B on V_i induces a linear action on quotient space V_i/V_{i-1}, or in other words, there is a group homomorphism

$$B \to GL(V_n/V_{n-1}) \times GL(V_{n-1}/V_{n-2}) \cdots \times GL(V_1/V_0).$$

定义 4 (Nilponent subgroup)**.** *Define U to be the kernel of the above homomorphism.*

1. *Prove that the action of G on FL is transitive and hence the Borel subgroups are conjugate to each other.*

2. *Restrict the action of G on FL to B. Find a bijection between the set of B-orbits with S_n, and for each orbit corresponding to $\omega \in S_n$, find a bijection to $F^{l(\omega)}$.*

3. *Prove that B is the normalizer of U in G.*

4. *Prove that U is a unipotent group.*

5. *Is it true that U is the commutator subgroup of B?*

6. Define a partial flag to be a chain of subspaces

$$F : \{0\} = V_0 \subsetneq V_1 \subsetneq V_2 \cdots \subsetneq V_m = V.$$

Find the relations between partial flags and parabolic subgroups.

题 11. *Let F be a field. Find the derived subgroup or the commutator subgroup of $GL(n, F)$.*