代数 1 H 班 作业 6

2022 年 10 月 20 日

题 1. *Prove there is a unique ring homomorphism* $\phi\colon \mathbb{Z} \to R$*. Assume* $\ker\phi = (m)$ *with* $m \geq 0$*. Then* m *is called the characteristic of* R*. Prove that the characteristic of an integral domain must be a prime number. Prove that the number of elements in a finite field must be a power of its characteristic.*

题 2 (Formal power series)**.** *Let* F *be a field. Define* $F[[x]]$ *to be the set of formal power series* $f(x) = a_0 + a_1x + a_2x^2 + \cdots$ *with* $a_i \in F$*. The ring structure on* $F[[x]]$ *is defined similarly as ring of polynomials.*

1. *Prove that* $F[[x]]$ *is a Euclidean domain with size function* $\sigma(f) = i$ *being the smallest degree of* x^i *with nonzero coefficient* a_i*.*

2. *Find the units in* $F[[x]]$*.*

3. *Artin Chapter 11, 3.10, Determine all ideals of the ring* $F[[t]]$ *of formal power series with coefficients in a field* F *(see Exercise 2.2).*

4. *Define* $F((x))$ *to be the set of Laurent series* $f(x) = \sum_{i\geq n} a_i x^i$ *where* n *varies in* $\mathbb{Z}$*. For example* $f(x) = -x^{-2} + x^{-1} + 1 + x + x^2 + x^3 + \cdots x^n + \cdots$*. The ring structure on* $F((x))$ *is defined similarly as polynomials. Prove that* $F((x))$ *is a field.*

题 3 (Artin Chapter 11, 3.8)**.** *Let* R *be a ring of prime characteristic* p*. Prove that the map* $R \to R$ *defined by* $x \rightsquigarrow x^p$ *is a ring homomorphism. (It is called the Frobenius map.)*

题 4 (Artin Chapter 11, 3.9)**.**

1. *An element* x *of a ring* R *is called nilpotent if some power is zero. Prove that if* x *is nilpotent, then* $1 + x$ *is a unit.*

2. *Suppose that R has prime characteristic $p \neq 0$. Prove that if a is nilpotent then $1+a$ is unipotent, that is, some power of $1+a$ is equal to 1.*

题 5 (Fractions)**.** *Let R be an integral domain. Consider a set $X = \{(a,b) | a, b \in R, b \neq 0\}$. Define a relation on X by $(a,b) \sim (c,d)$ if and only if $ad = bc$.*

1. *Prove that this is an equivalence relation.*
2. *Denote by $\frac{a}{b}$ the equivalence class containing (a,b) and it is called a fraction. The set of fractions is denoted by $Frac(R)$ and there is a map $f\colon R \to Frac(R)$ defined by $a \mapsto \frac{a}{1}$. Try to define a field structure on F such that the map f is an injective ring homomorphism. This is called the fraction field of R.*
3. *Prove that if there is an injective ring homomorphism $R \to F$ with F being a field. Then this homomorphism factors through $R \to Frac(R) \to F$.*
4. *Let F be a field. Prove that $Frac(F[[x]])$ is isomorphic to $F((x))$.*

题 6 (Artin Chapter 11, 4.3)**.** *Try to identify the following rings with forms you think simpler.*

1. $\mathbb{Z}[x]/\left(x^2-3, 2x+4\right)$,
2. $\mathbb{Z}[i]/(2+i)$,
3. $\mathbb{Z}[x]/(6, 2x-1)$
4. $\mathbb{Z}[x]/\left(2x^2-4, 4x-5\right)$
5. $\mathbb{Z}[x]/\left(x^2+3, 5\right)$.

题 7. *Classify all the maximal ideals and prime ideals of $\mathbb{R}[x]$.*

题 8. *Prove that the ring $\mathbb{Z}[\omega]$ is a Euclidean domain. Here $\omega = e^{\frac{2\pi\sqrt{-1}}{3}}$.*

题 9. *Prove that the ring $\mathbb{Z}[\sqrt{-2}]$ is a Euclidean domain, hence PID.*

题 10. *Prove that $\mathbb{C}[x,y]$ is not a PID.*

题 11 (Product ring)**.** *Let R and R' be rings. There is a ring structure on the product $R \times R'$ defined by $(a, a') + (b, b') = (a + b, a' + b')$ and $(a, a')(b, b') = (ab, a'b')$. This is called the product ring $R \times R'$.*

1. *What is the multiplicative identity in product ring $R \times R'$?*

2. *An idempotent element e of a ring S is an element in S such that $e^2 = e$. Prove that the principal ideal (e) is itself a ring with identity element e.*

3. *If e is an idempotent element, then $1-e$ is also an idempotent element.*

4. *Prove that $(1, 0) \in R \times R'$ is an idempotent element.*

5. *If e is an idempotent element, Prove that S is isomorphic to the the product ring $(e) \times (1 - e)$.*

6. *Prove that there is a bijection between the set of prime ideals in $R \times R'$ and disjoint union of prime ideals in R and prime ideals in R'.*

题 12. *Artin, Chapter 11, 6.8* *Let I and J be ideals of a ring R such that $I + J = R$.*

1. *Prove that $IJ = I \cap J$.*

2. *Prove the Chinese Remainder Theorem: For any pair a, b of elements of R, there is an element x such that $x \equiv a \mod I$ and $x \equiv b \mod J$. (Here the notation $x \equiv a \mod I$ means $x - a \in I$.*

3. *Prove that if $IJ = 0$, then R is isomorphic to the product ring $R/I \times R/J$. (Hint: consider the natural homomorphism from $R \to R/I \times R/J$.*

4. *Describe the idempotents corresponding to the product decomposition in (c).*

题 13. *Artin Chapter 11, 5.1* *Let $f = x^4+x^3+x^2+x+1$ and let α denote the residue of x in the ring $R = \mathbb{Z}[x]/(f)$. Express $(\alpha^3 + \alpha^2 + \alpha)(\alpha^5 + 1)$ in terms of the basis $(1, \alpha, \alpha^2, \alpha^3)$ of R.*

题 14. *Use* $Hibert'sNullstellensatz$ *to determine all the maximal ideals in* $\mathbb{C}[x,y]/(xy)$.

题 15. *Let* $\mathbb{Z}[\sqrt{-3}] = \{m+n\sqrt{-3}|m,n \in \mathbb{Z}\}$. *Is* $(\sqrt{-3}+2)$ *a maximal ideal in* $\mathbb{Z}[\sqrt{-3}]$*? Why?*

题 16. ***Artin Chapter 11, 8.3*** *Prove that* $\mathbb{F}_2[x]/(x^3+x+1)$ *is a field, but* $\mathbb{F}_3[x]/(x^3+x+1)$ *is not a field.*

题 17 (Artin Chapter 11, 9.13)**.** *Let* $\psi\colon \mathbb{C}[x,y] \to \mathbb{C}[t]$ *be a homomorphism that is identity on* $\mathbb{C}$ *and sends* $x \mapsto x(t)$, $y \mapsto y(t)$ *such that* $x(t)$ *and* $y(t)$ *are not both constant. Prove that kernel of* ψ *is principal ideal.*

题 18 (Artin Chapter 11, M.7)**.** *Let* X *denote the closed unit interval* $[0,1]$, *and let* R *be the ring of continuous functions* $X \to \mathbb{R}$.

1. *Let* $f_1, \ldots, f_n$ *be functions with no common zero on* X. *Prove that the ideal generated by these functions is the unit ideal. Hint: Consider* $f_1^2 + \cdots + f_n^2$.

2. *Establish a bijective correspondence between maximal ideals of* R *and points on the interval.*