代数 1 H 班 作业 7

2022 年 11 月 9 日

题 1. *Prove that* $\mathbb{Z}[i]/(3)$ *is a field.*

题 2. *Give an example of irreducible polynomial* $f(x)$ *of degree* 2 *in* $\mathbb{F}_3[x]$*. Use* $f(x)$ *to construct an example of a field consisting of* 9 *elements.*

题 3. *Decide whether or not* x^4+6x^3+9x+3 *is irreducible in* $\mathbb{Q}[x]$*.*

题 4. *Factor the integral polynomial* $x^5+2x^4+3x^3+3x+5$ *in* $\mathbb{F}_2[x]$*,* $\mathbb{F}_3[x]$ *and* $\mathbb{Q}[x]$*.*

题 5 (Artin Chapter 12, 2.9)**.** *Let* F *be a field. Prove that the ring* $F[x,x^{-1}]$ *of Laurent polynomials (Chapter 11 , Exercise 5.7) is a principal ideal domain. (Hint: use ring homomorphism* $\phi\colon F[x]\to F[x,x^{-1}]$ *and pull-back of ideals.)*

题 6. 假设 $x\in\mathbb{Q}$ 是某一个首一整系数多项式的根，则 x 是整数。

题 7. 假设 D 是一个正整数.

1. 验证 $\mathbb{Z}[\sqrt{-D}]=\{a+b\sqrt{-D}\mid a,b\in\mathbb{Z}\}$ 是一个 $\mathbb{C}$ 的子环.
2. 求 $\mathbb{Z}[\sqrt{-D}]$ 的乘法可逆元全体.
3. 在 $\mathbb{Z}[\sqrt{-5}]$ 中验证 $2,3,1\pm\sqrt{-5}$ 是不可约元。请指出哪些是素元，为什么？
4. 证明 $\mathbb{Z}[\sqrt{-D}]$ 中的非零素理想都是极大理想。*(Hint: Use ring homomorphism* $\mathbb{Z}\to\mathbb{Z}[\sqrt{-D}]$ *and pull back of ideals.)*

题 8. *Prove that a prime number* p *can be written as* $p = m^2 + 2n^2$ *with* $m, n \in \mathbb{Z}$ *if and only if* $x^2 + 2$ *has a root in* $\mathbb{F}_p$. *(In fact, this is true if and only if* $p = 2$ *or* $p \equiv 1, 3 \mod 8$*, proved by Fermat.)*

题 9. 求一组多项式方程 $f^2+g^2 = h^2$ 在 $\mathbb{C}[t]$ 上的非平凡解，即 $f, g, h \in \mathbb{C}[t]$ 满足 $\deg f, \deg g, \deg h \geq 1$ 且 $\gcd(f, g, h) = 1$.（所有解可以是什么形式？)

题 10. 尝试描述 $\mathbb{C}[x, y]$ 的所有素理想。

题 11. *Let* F *be a field and* $f(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots a_0 \in F[x]$. *Define the derivative of* f *similarly as calculus.* $f'(x) = na_n x^{n-1} + (n-1)a_{n-1}x^{n-2} + \cdots + a_1$. *For* $f(x), g(x) \in F[x]$*, prove*

1. $(fg)' = fg' + f'g$.
2. $(f(g(x)))' = f'(g(x)) \cdot g'$.
3. $\gcd(f, f') = 1$ *if and only if in the irreducible factorization of* f*, there are no factors with multiplicities.*

题 12. *Let* p *be a prime number. Prove that* $f(x) = x^p - x - 1 \in \mathbb{F}_p[x]$ *is irreducible by the following steps.*

1. $f(x + 1) = f(x)$
2. *If* $g(x) \in \mathbb{F}_p[x]$ *and* $1 \leq \deg g \leq p - 1$*, then* $g(x + 1) \neq g(x)$.
3. *Let* $f = g_1 \cdots g_k$ *be irreducible factorization of* f *with monic factors. Prove that* g_i *are distinct.*
4. *Let* $a \in \mathbb{Z}/p\mathbb{Z}$. *Prove that* $a \cdot g_i(x) = g_i(x + a)$ *defines an action of* C_p *on the set* $\{g_1 \cdots g_k\}$.
5. *Using enumeration formulas of group operation to prove that* $f(x)$ *is irreducible.*

(In fact, for $n \geq 2$*, the polynomial* $f(x) = x^n - x - 1 \in \mathbb{Z}[x]$ *is irreducible.)*