代数 2 H 班 作业 1

2023 年 8 月 2 日

题 1. *Prove that the polynomial* $x^4 + 3x + 3$ *is irreducible polynomial over the field* $\mathbb{Q}[\sqrt[3]{2}]$.

题 2. *Find the degree of field extension*

1. $[\mathbb{Q}[\sqrt{p}, \sqrt{q}] \colon \mathbb{Q}]$ *where* p *and* q *are two distinct prime numbers.*
2. $[\mathbb{Q}[\sqrt[3]{2}, \sqrt{2}] \colon \mathbb{Q}]$

题 3. *Find the irreducible polynomial of* $\sqrt[3]{2} + \sqrt{3}$ *over* $\mathbb{Q}$.

题 4. *We call two extensions* K_1 *and* K_2 *of* F *isomorphic if there exists a field isomorphism* $\varphi \colon K_1 \to K_2$ *such that* $\varphi|_F \colon F \to F$ *is identity. Classify the isomorphism classes of degree-two extensions (quadratic extensions) of* $\mathbb{Q}$.

题 5. *Consider quadratic extensions* K *of* F *with* $\operatorname{char} F = 2$. *Prove that either* $K = F[\alpha]$ *with* $\alpha^2 \in F$ *and* $\alpha \notin F$ *or* $K = F[\alpha]$ *with* $\alpha^2 - \alpha \in F$ *and* $\alpha \notin F$. *Can two cases of two different types above be isomorphic?*

题 6. *Classify degree-two extensions of* $\mathbb{F}_2(x)$.

题 7. *Determine whether a regular* 9*-gon is constructible or not by ruler and compass.*

题 8. *Find the degree* $[K : F]$ *of the splitting field* K *of* $f(x)$ *over* F.

1. $F = \mathbb{Q}$, $f(x) = x^5 - 2$
2. $F = \mathbb{F}_p$, $f(x) = x^p - x - 1$.

题 9. *Let K be the splitting field of a degree-n polynomial $f(x)$ over F. Prove that the degree $[K : F]$ divides $n!$. Can you find cases such that $[K : F] = n!$ for each integer n?*

题 10. *Determine whether the following three fields are isomorphic.*

1. *The splitting field of $x^2 - t^3$ over $\mathbb{Q}(t)$.*
2. *The splitting field of $x^2 - t^5$ over $\mathbb{Q}(t)$.*
3. *The splitting field of $x^2 + t^2$ over $\mathbb{Q}(t)$.*