代数 2 H 班 作业 10

2023 年 8 月 2 日

We always denote by R a commutative ring.

题 1. *Define the dimension of a topological space X as follows. Let $\phi \neq X_0 \subsetneq X_1 \subsetneq X_2 \cdots \subsetneq X_n$ be a chain of irreducible closed subsets of X. We call n the length of this chain. Then $\dim X$ is defined to be the maximal length of such chains of irreducible closed subsets in X. Let $X = \mathrm{Spec}(R)$, and define $\dim R = \dim \mathrm{Spec}\, R$. Please write down a definition of for dimension $\dim X$ using prime ideals of R. Find $\dim R$ for PID R. Calculate $\dim \mathbb{C}[x, y]$.*

题 2. *In last homework, you showed that there is an one-to-one correspondence between radical ideals of R and closed subsets of $\mathrm{Spec}(R)$ by $I \mapsto Z(I)$. Under this correspondence, describe the operation on ideals when we take union of finitely many close subsets.*

题 3.

1. *Prove that R has a unique maximal ideal m if and only if $R\backslash m$ is the set of units of R. (We call such R a local ring.)*
2. *Let p be a prime ideal of R. Prove that R_p is a local ring with maximal ideal pR_p and R_p/pR_p is isomorphic to the fractional field of R/p.*

题 4. *A&M chapter 3, ex 3.*

题 5. *A&M chapter 3, ex 7.*

题 6. *A&M chapter 3, ex 8.*

题 7.

1. *Consider $\mathrm{Spec}(R)$ and $f, g \in R$. Show that if open subsets $U_f \subset U_g$, there is a natural restriction map $R_g = R[g^{-1}] \to R[f^{-1}] = R_f$. Especially when $U_f = U_g$, prove that $R_g \cong R_f$.*

2. *Let p be a prime ideal of R. We have the following description of local ring R_p. Consider all $f \in R$ such that $p \in U_f$. We define an equivalence relation on the disjoint union R_f, ($p \in U_f$) by $a \in R_f \sim b \in R_g$ iff there is open subset $U_h \subset U_g \cap U_f$ and $p \in U_h$, such that the restrictions of a and b in R_h are equal. Show that there is a natural ring structure on such equivalence classes and it is isomorphic to R_p. (We call R_p the "germ" of "functions" at p, or the stalk of the structure sheaf defined by R.)*

题 8. *Assume $f_1, \cdots, f_n, f \in R$. Assume $U_f = \bigcup_{i=1}^n U_{f_i}$. We still use f_i to denote the image of f_i in R_f under the restriction map $R \to R_f$. Prove that $R_f[f_i^{-1}] \cong R_{f_i}$.*

Use this to show the sheaf property for R on open set U_f. In other words, let $\phi_i \colon R_f \to R_{f_i}$, $\phi_{ij} \colon R_{f_j} \to R_{f_i f_j}$ be the restriction maps. The sequence

$$0 \to R_f \to \prod_{i=1}^n R_{f_i} \to \prod_{1 \le i,j \le n} R_{f_i f_j}$$

is exact. The map from $R_f \to \prod_{i=1}^n R_{f_i}$ is the restriction map $a \mapsto (\phi_i(a))_i$. And $R_{f_i} \to \prod_{1 \le i,j \le n} R_{f_i f_j}$ is defined by $(a_i)_i \mapsto (\phi_{ij}(a_j) - \phi_{ji}(a_i))_{ij}$.

题 9 (选做)**.** *State and prove the sheaf property for R-module M and find the stalk of this sheaf at $p \in \mathrm{Spec}(R)$.*

题 10. *A sequence of morphisms among R-modules $A_1 \xrightarrow{f_1} A_2 \xrightarrow{f_2} A_3 \xrightarrow{f_3} A_4 \cdots A_n$ is said to be exact if $\operatorname{im} f_i = \ker f_{i-1}$. Consider the sequence $0 \to A \xrightarrow{f} B \xrightarrow{g} C$.*

1. *Prove that this sequence is exat if and only if A is isomorphic to $\ker g$ under the map f.*

2. *Prove the following universal property of exact sequence $0 \to A \xrightarrow{f} B \xrightarrow{g} C$. If there is a morphism $h \colon P \to B$ between R-modules and $g \circ h = 0$, then there exists a unique morphism $\bar{h} \colon P \to A$, such that $h = f \circ \bar{h}$. Conversely, if the sequence satisfies this property, then it is exact.*

3. *For a morphism* $g\colon B \to C$*, construct a natural bijection between* $Hom_R(P, \ker g)$ *and subset of* $Hom_R(P, B)$ *defined by*

$$\{h \in Hom_R(P, B) \mid g \circ h = 0\}$$

4. *Let* $f\colon A \to B$ *be morphism between two* R*-modules. Define* $\operatorname{coker} f = B/\operatorname{im} f$*. Construct a natural bijection between* $Hom_R(\operatorname{coker} f, P)$ *and subset of* $Hom_R(B, P)$ *defined by*

$$\{h \in Hom_R(B, P) \mid h \circ f = 0\}$$