代数 2 H 班 作业 11

2023 年 8 月 2 日

We always denote by R a commutative ring.

题 1. *Read Eisenbud Appendix A5.1, A5.3 and definition of ajoints in the introduction part of A5.2*

题 2. *Calculate* $\mathbb{Z}/n\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z}/m\mathbb{Z}$ *for positive integers* m *and* n.

题 3. *Let* V *be a free* R*-module with basis* $x, x \in X$ *and* W *a free* R*-module with basis* $y, y \in Y$. *Show that the tensor product of* V *and* W *is free with basis* $x \otimes y$. *(Try to think about how to view this from universal properties of tensor product and free modules.)*

题 4. *Let* M *be a* R *module. Prove that both* $Hom_R(-, M)$ *and* $Hom_R(M, -)$ *are left exact. In other words, If*

$$0 \to A \to B \to C$$

is exact, then the induced sequence

$$0 \to Hom_R(M, A) \to Hom_R(M, B) \to Hom_R(M, C)$$

is exact.

If

$$A \to B \to C \to 0$$

is exact, then the induced sequence

$$0 \to Hom_R(C, M) \to Hom_R(B, M) \to Hom_R(A, M)$$

is exact. (It is not mandatory, but it is recommended to use the adjoint property to prove this)

题 5. *In general, tensor product does not commute with direct product. Consider $\mathbb{Z}$ module $\prod_{n\geq 1}\mathbb{Z}/n\mathbb{Z}$. Construct a non-torsion element in $\prod_{n\geq 1}\mathbb{Z}/n\mathbb{Z}$. Show that $(\prod_{n\geq 1}\mathbb{Z}/n\mathbb{Z})\otimes_{\mathbb{Z}}\mathbb{Q}$ is not equal to zero and $\prod_{n\geq 1}(\mathbb{Z}/n\mathbb{Z}\otimes_{\mathbb{Z}}\mathbb{Q})=0$. (You can use the theorem about localization of a module and tensor product $M[S^{-1}]\cong M\otimes_R R[S^{-1}]$, it will be proved later in class)*

题 6. *Let A and B be two R algebras. Let $\pi_1\colon A\to A\otimes_R B, a\mapsto a\otimes 1$ and $\pi_2\colon B\to A\otimes_R B, b\mapsto 1\otimes b$ be two homomorphisms of R-algebras. Show the universal property of $A\otimes_R B$. In other words, if there is a R-algebra C with $f_1\colon A\to C$ and $f_2\colon B\to C$, then there exists a unique homomorphism of R-algebra $f\colon A\otimes_R B\to C$ such that $f_i=f\circ\pi_i$.*

题 7. *Simplify $\mathbb{C}[t]\otimes_{\mathbb{C}}\mathbb{C}[t]$, $\mathbb{C}[t]\otimes_{\mathbb{C}[t]}\mathbb{C}[t]$ and $\mathbb{C}[t,s]\otimes_{\mathbb{C}[t]}\mathbb{C}[t,s]$. Here $\mathbb{C}[t]$ and $\mathbb{C}[t,s]$ are $\mathbb{C}[t]$-modules via the natural embedding.*

题 8. *Here we introduce another definition of tensor product and prove that it is the same as the one defined in class.*

Let M and N be two R-modules and G be an abelian group. We call a map $f\colon M\times N\to G$ "R-balanced" if the map is $\mathbb{Z}$-bilinear and also satisfies $f(rm,n)=f(m,rn)$ for any $r\in R$, $m\in M$ and $n\in N$. The set of such maps is denoted by $Hom_{R-balance}(M\times N,G)$.

1. *Show that there is a bijection between*

$$Hom_{R-balance}(M\times N,G)\cong Hom_R(M,Hom_{\mathbb{Z}}(N,G))$$

Here the R-module structure on $Hom_{\mathbb{Z}}(N,G)$ is given by $(r\phi)(n)=\phi(rn)$ for any $\phi\in Hom_{\mathbb{Z}}(N,G)$.

2. *Construct an abelian group $M\tilde{\otimes}N$ such that there is an natural bijection between*

$$Hom_{\mathbb{Z}}(M\tilde{\otimes}N,G)\cong Hom_R(M,Hom_{\mathbb{Z}}(N,G)).$$

Try to write it as quotient group of free abelian group with basis $M\times N$ quotient by some relations. Denote by $m\tilde{\otimes}n$ for the image of $(m,n)\in M\times N$ in $M\tilde{\otimes}N$. State the universal property of $M\tilde{\otimes}N$.

3. *Use the universal property to prove that* $r \cdot m\tilde{\otimes}n = (rm)\tilde{\otimes}n$ *gives a well defined* R*-module structure on* $M\tilde{\otimes}N$. *Prove that the natural map* $M \times N \to M\tilde{\otimes}N$ *is* R *bilinear under this* R*-module structure.*

4. *Show that* $M\tilde{\otimes}N \cong M \otimes N$ *as* R*-module.*

   (Hint, use the map $M \times N \to M\tilde{\otimes}N$ *and universal property of* $M\otimes N$ *to construct a map* $F\colon M \otimes N \to M\tilde{\otimes}N$ *and vice versa)*