代数 2 H 班 作业 12

2023 年 8 月 2 日

We always denote by R a commutative ring.

题 1. *Let M be finitely generated R-module. Prove that any surjective R-module morphism $\phi\colon M \to M$ is an isomorphism. Does similar property holds for injective morphism? (Hint: consider the $R[t]$-module defined by ϕ and apply Caley-Hamilton theorem for $I = (x)$ and identity operator.)*

题 2. *A&M Chapter 2, Exercise 3*

题 3. *A&M Chapter 2, Exercise 4*

题 4. *A&M Chapter 2, Exercise 10*

题 5. *A&M Chapter 2, Exercise 11*

题 6. *Assume M is a flat R-module. Prove that for any exact sequence $N_1 \to N_2 \to \cdots \to N_m$ of R-modules,the corresponding sequence $M \otimes N_1 \to M \otimes N_2 \to \cdots \to M \otimes N_m$ is exact. (Hint: use kernel, image, cokernel to decompose the exact sequence into short exact sequences.)*

题 7. *Let $0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0$ be short exact sequence of R-modules. Prove that the following are equivalent*

1. *There is homomorphism between R-modules $h\colon B \to A$ such that $h \circ f = id_A$.*

2. *There is homomorphism between R-modules $p\colon C \to B$ such that $g \circ p = id_C$.*

3. *$f(A)$ has a complement in B, in other words, there is a submodule of C' of B, such that B is the direct sum of $f(A)$ and C'.*

If one of the conditions holds, we call such a short exact sequence splits. Find an example of non-split short exact sequence of $\mathbb{Z}$-modules.

题 8. *We define a R-module P to be projective if and only if $Hom_R(P, -)$ is exact. In other words, for any short exact sequence $0 \to A \to B \to C \to 0$ of R-modules, the corresponding sequence*

$$0 \to Hom_R(P, A) \to Hom_R(P, B) \to Hom_R(P, C) \to 0$$

is exact. Prove that P is projective if and only if it is a direct summand of a free module. (Hint: if P is projective, construct a surjective morphism from free module F to P, show that $0 \to \ker f \to F \to P \to 0$ splits.) Prove that any projective module P is flat.

题 9. *Use Nakayama's lemma to prove that if R is a local ring with maximal ideal m, and M is a finitely generated projective R-module, then M is free.*

1. *Let $x_1 \cdots x_n \in M$ form a R/m basis for $M \otimes R/m$. Prove that $x_1 \cdots x_n$ are also generators for M.*

2. *Use $x_1 \cdots x_n \in M$ to construct a surjective morphism $R^n \to M$. Prove that $R^n \cong M \oplus N$ for some module N.*

3. *Prove that $N = 0$.*

题 10. *Let R be a noetherian ring and M is a finitely generated projective module. Prove that*

1. *M_p is projective R_p-module.*

2. *M_p is free R_p-module (This is called locally free).*

3. *There exists $f_1 \cdots f_n \in R$ such that U_{f_i} covers $\operatorname{Spec} R$, and each localization M_{f_i} is free R_{f_i}-module. (This is called locally free in the strong sense.)*