代数 2 H 班 作业 13

2023 年 8 月 2 日

We always denote by R a commutative ring.

题 1. *Let R be a UFD, prove that R is normal (integral closed in its fractional field.)*

Integrality has many geometric propositions. In the following, we state and prove these properties. Recall we define the Krull dimension of a topological space X using irreducible closed subsets. Let $\phi \neq X_0 \subsetneq X_1 \subsetneq \cdots \subsetneq X_n$ be a chain of irreducible closed subsets of X. Then $\dim_{\text{Krull}} X$ is defined to be the supremum of length n of such chains. For any commutative ring R, we define $\dim_{\text{Krull}} R = \dim_{\text{Krull}} \operatorname{Spec}(R)$. From the correspondence between irreducible closed subsets and prime ideals, we know that $\dim_{\text{Krull}} R = \sup\{n | p_0 \subsetneq p_1 \subsetneq \cdots \subsetneq p_n, p_i \in \operatorname{Spec} R\}$. Two basic building blocks are important to us, $\dim_{\text{Krull}} k = 0$ for a field k and $\dim_{\text{Krull}} R = 0, 1$ for a PID R.

题 2. *If $R \to A$ is an integral ring homomorphism, prove that $\operatorname{Spec} A \to \operatorname{Spec} R$ is a closed mapping, i.e. the image of a close subset is closed. (Reduce to integral ring extension and use going-up)*

题 3. *Use "Going-up" and "Imcomparability" to prove that if $R \subset A$ be an integral ring extension, then $\dim_{\text{Krull}} A = \dim_{\text{Krull}} R$. (How about the case of integral ring homomorphism?)*

We say a ring homomorphism $A \to B$ is finite if B is a finite A-module via this ring homomorphism. Or we can also say B is finite over A. In particular, any finite ring homomorphism is integral.

题 4. *Let $A \to B \to C$ be ring homomorphisms. Show that if B is finite over A and C is finite over B, then C is finite over A.*

题 5. *Let $A \to B$ be ring homomorphism and B is a finitely generated A-algebra under this ring homomorphism. If B is integral over A, prove that B is finite over A.*

题 6. *Let k be a field with infinitely many elements. Let $B = k[y_1, \cdots, y_m]/J$ be a finitely generated k-algebra and $J \neq 0$. Prove that there are $m-1$ k-linear combinations of $y_1 \cdots y_m$, denoted by $z_1, \cdots, z_{m-1}$ such that B is finite over the k-subalgebra generated by $z_1, \cdots, z_{m-1}$. (See A&M chapter 5, exercise 16 for more hint on the construction, assume there exists a polynomial $f \in J \neq 0$ in n variables such that $f(y_1, \ldots, y_{m-1}, y_m) = 0$. Let F be the homogeneous part of highest degree in f. Since k is infinite, there exist $\lambda_1, \ldots, \lambda_{m-1} \in k$ such that $F(\lambda_1, \ldots, \lambda_{m-1}, 1) \neq 0$. Put $y_i' = y_i - \lambda_1 y_m$ $(1 \leqslant i \leqslant m-1)$. Show that y_m is integral over the ring $A' = k\left[y_1', \ldots, y_{m-1}'\right]$)*

题 7 (Noether normalization)**.** *Let k be a field with infinitely many elements and $A = k[x_1 \cdots x_n]/I$ is a finitely generated k-algebra. Prove that there exist k-linear combinations of $x_1 \cdots x_n$, denoted by $y_1, \cdots, y_m$ such that the ring homomorphism $R = k[t_1 \cdots t_m] \to A, t_i \mapsto y_i$ is injective and finite (hence an integral ring extension). For example, $A = k[x,y]/(xy)$ is an integral ring extension over $R = k[x+y]$. (Geometrically speaking, there is a linear subspace V of k^n such that the algebraic variety defined by I is mapped "nicely" (surjective closed mapping with finite fiber) onto V.) (Hint: Let m be the minimal number such that there exists $y_1, \cdots, y_m$ such that $R = k[t_1 \cdots t_m] \to A, t_i \mapsto y_i$ is an integral ring homomorphism. First show this minimal value exists. Then show that this ring homomorphism is injective.)*

This conclusion is still true for k finite field and y_i allowing to be polynomials of $x_1, \cdots, x_n$, if you are interested, see Eisenbud page 282, lemma 13.2. The key trick is due to Nagata.

题 8. *Let k be a field with infinitely many elements. Show that $\dim_{\mathrm{Krull}} k[x_1, \cdots, x_n] = n$. (Use induction and previous conclusions in the homework)*

题 9. *Let* $\phi\colon R \to A$ *be a finite ring homomorphism. Prove that* $\operatorname{Spec} A \to \operatorname{Spec} R$ *has finite fibers. In other words, for any* $p \in \operatorname{Spec} R$*, there are only finitely many* $q \in \operatorname{Spec} A$ *such that* $\phi^{-1}(q) = p$*.*

1. *Reduce the question to finite ring extension, i.e.* ϕ *injective.*
2. *Use localization to reduce this to* R *a local ring with maximal ideal* p*.*
3. *Let* $k = R/p$ *be the quotient field. Prove that the tensor product of* R*-algebras* $A \otimes_R k$ *is a finite-dimensional* k*-vector space.*
4. *Prove that* $\operatorname{Spec} A \otimes_R k$ *has Krull-dimension zero and has only finitely many maximal ideals. (Hint: it is integral over* k *and Noetherian.)*
5. *Prove that there is a one-to-one correspondence between preimages of* p *in* $\operatorname{Spec} A$ *and* $\operatorname{Spec} A \otimes_R k$*. (Hint: use the universal property of* R*-tensor product. The ring homomorphism* $A \to A/q$ *and* $R/p \to A/q$ *induce a ring homomorphism* $A \otimes_R k \to A/q$ *and vice versa.)*

This proof uses the idea of "fiber product" in algebraic geometry. The underlying tool is tensor product of algebras.

See A&M chapter 5, exercises 12, 13, 14, 15 for counting of fibers in Field and Galois theory.