代数 2 H 班 作业 2

2023 年 8 月 2 日

题 1 (Artin 15.7.14)**.** *Count the number of irreducible polynomials in* $\mathbb{F}_p[x]$ *of degree* n*.*

题 2 (Artin 15.7.3)**.** *Find a* $13th$ *root of* 2 *in* $\mathbb{F}_{13}$*.*

题 3. *Let* F *be a finite field. Prove that any irreducible polynomial* $f(x) \in F[x]$ *over a finite field has no multiple roots in any extension* K *of* F*.*

题 4. *Let* $f(x) \in \mathbb{C}((x))$*. Prove that there is a solution to equation* $y^n - f(x) = 0$ *for* $y \in \mathbb{C}((x^*))$*.*

题 5 (Artin 15.M.1)**.** *Let* $K = F(\alpha)$ *be a field extension generated by a transcendental element* α*, and let* β *be an element of* K *that is not in* F*. Prove that* α *is algebraic over* $F(\beta)$*.*

题 6. *Let* $F \subset K \subset L$ *be field extensions. Let* $\alpha_1, \cdots, \alpha_n \in L$*. Assume* K *is algebraic over* F*. Prove that* $K(\alpha_1, \cdots, \alpha_n)$ *is algebraic over* $F(\alpha_1, \cdots, \alpha_n)$*.*

题 7 (Lang V.26)**.** *Let* k *be a field,* $f(x)$ *an irreducible polynomial in* $k[x]$*, and let* K *be a finite normal extension of* k*. If* g, h *are monic irreducible factors of* $f(x)$ *in* $K[x]$*. Show that there exists an automorphism* σ *of* K *over* k *(i.e. element in* $\mathrm{Aut}_k(K)$*) such that* $\sigma(f) = g$*. Give an example when this conclusion is not valid if* K *is not normal over* k*.*

题 8 (思考题)**.** *Let* $Q(x) \in \mathbb{C}(x)$ *be a non-constant rational function. Find the degree of field extension* $[\mathbb{C}(x) : \mathbb{C}(Q(x))]$*.*

题 9. *Let* K/F *be a finite extension of fields. Prove that* K/F *is normal if and only if for any irreducible polynomial* $f(x) \in F[x]$*, the irreducible factors of* $f(x)$ *in* $K[x]$ *have the same degree.*

题 10. *Let K and L be two extensions of F and K is a normal extension. Prove that the extension generated by K and L is "well-defined", in other words, independent from the common extension for K and L. Show an example that this fails when K is not normal.*

题 11. *Let K be a normal extension of F and L be an intermediate extension $F \subset L \subset K$. Show that any F-map from L to K extends to K.*