代数 2 H 班 作业 3

2023 年 8 月 2 日

题 1. *Let K be an algebraic extension of F and K_s be the intermediate extension consisting of separable elements over F. Prove the following*

1. *The extension K_s/F is separable.*
2. *The extension K/K_s is pure inseparable, i.e. for all $\alpha \in K$, there exists positive integer e such that $\alpha^{p^e} \in K_s$.*
3. *If K/F is finite, then $|Hom_F(K, \bar{F})| = [K_s : F]$.*
4. *If K/F is normal, prove that K_s/F is normal.*

题 2. *Prove that for simple algebraic extension $F[\gamma]$ of F, there are only finitely many intermediate extension $F \subset L \subset F[\gamma]$.*

题 3. *Let k be a field with characteristic $p \neq 0$. Let $K = k(t, u)$ and $F = k[t^p, u^p]$. Prove that if $F[t + au] = F[t + bu]$ for $a, b \in k$, then $a = b$. Hence if k is an infinite field, there are infinite intermediate extensions $F \subset L \subset K$.*

题 4. *Let F be a field. Prove that*

$$Aut_F(F(x)) = \{x \mapsto \frac{ax+b}{cx+d} \mid \det \begin{bmatrix} a & b \\ c & d \end{bmatrix} \neq 0\}$$

and $Aut_F(F(x)) \cong PGL(2, F)$ as groups.

题 5. *Let $\sigma \colon x \mapsto \frac{1}{x}$ and $\tau \colon x \mapsto e^{\frac{2\pi\sqrt{-1}}{n}} x$ be automorphisms in $Aut_{\mathbb{C}}(\mathbb{C}(x))$. Prove that the group H generated by σ and τ is finite and identify this group with finite groups you know. Find the fixed field $\mathbb{C}(x)^H$.*

题 6. *Find a subgroup of* $Aut_F F(x_1, \cdots, x_n)$ *which is isomorphic to* $PGL(n+1, F)$ *and find an element not in this subgroup when* $n \geq 2$*.*

题 7. *Prove that* F *being perfect is equivalent to that all finite extensions of* F *are separable.*

题 8. *Let* K/F *be a finite extension. Prove that* $Aut_F(K)$ *divides* $[K : F]$*.*

题 9. *Let* F *be a field with finite characteristic* p *and finite extension* K/F*. Denote by* K^p *be the image of Frobenius morphism of* K*. Prove that* K/F *being separable is equivalent to* $FK^p = K$*.*