代数 2 H 班 作业 4

2023 年 8 月 2 日

题 1. *Let E and L be two finite extensions of F in a common extension. Is it always true that $[EL : L] = [E, E \cap L]$? If not, please provide a counter-example.*

题 2. *Find all the subextensions of $\mathbb{Q}[e^{\frac{2\pi\sqrt{-1}}{17}}]/\mathbb{Q}$. Try to find the primitive element in each subextension. (Optional question: write $\cos\frac{2\pi}{17}$ by formula involving only rational numbers, $+, -, \times, \div$ and square root.)*

题 3. *Find the Galois group of polynomial $x^4 - 2$ over $\mathbb{Q}$. (you can write your answer as generators and relations of the group, or the action on roots, or identity it with groups you know)*

题 4. *Let F be a field of characteristic $p > 0$. For any $a \in F$, the polynomial $x^p - x - a \in F[x]$ either splits in F or irreducible. Compute the Galois group of this polynomial over F. (Conversely, if E/F is a Galois extension of degree p, E is a splitting field of $x^p - x - a$ for some $a \in E$. We will prove this later).*

题 5. *Find the intermediate fields between $\mathbb{C}(t)$ and $\mathbb{C}(t^n + \frac{1}{t^n})$. Try to find the primitive element in each subextension.*

题 6. *Given a finite Galois extension K/F, consider a subextension L. We have the following two constructions to find a normal extension related to L/F. One is to take the minimal normal extension over F containing L, (called normal closure). The other is to find the smallest subextension E of F such that L/E is normal. Explain the corresponding operations on groups under Galois correspondence.*

题 7 (Milner's book Fields and Galois Theory)**.** *Let F be a field of characteristic 0. Show that $F\left(X^{2}\right) \cap F\left(X^{2}-X\right)=F$ (intersection inside $F(X)$). [Hint: Find automorphisms σ and τ of $F(X)$, each of order 2, fixing $F\left(X^{2}\right)$ and $F\left(X^{2}-X\right)$ respectively, and show that $\sigma\tau$ has infinite order.]*

题 8 (Milner's book Fields and Galois Theory)**.** *Let p be an odd prime, and let ζ be a primitive pth root of 1 in $\mathbb{C}$. Let $E=\mathbb{Q}[\zeta]$, and let $G=\operatorname{Gal}(E/\mathbb{Q})$; thus $G=(\mathbb{Z}/(p))^{\times}$. Let H be the subgroup of index 2 in G. Put $\alpha=\sum_{i \in H} \zeta^{i}$ and $\beta=\sum_{i \in G \backslash H} \zeta^{i}$. Show:*

1. *α and β are fixed by H;*
2. *if $\sigma \in G \backslash H$, then $\sigma \alpha=\beta, \sigma \beta=\alpha$.*

Thus α and β are roots of the polynomial $X^{2}+X+\alpha \beta \in \mathbb{Q}[X]$. Compute $\alpha\beta$ (or $\alpha-\beta$) and show that the fixed field of H is $\mathbb{Q}[\sqrt{p}]$ when $p \equiv 1 \bmod 4$ and $\mathbb{Q}[\sqrt{-p}]$ when $p \equiv 3 \bmod 4$.

题 9 (丘赛题目)**.** *Let p be a prime number, and let $\mathbb{F}_{p}$ be the finite field with p elements. Let $F=\mathbb{F}_{p}(t)$ be the field of rational functions over $\mathbb{F}_{p}$. Consider all subfields C of F such that F/C is a finite Galois extension.*

1. *Show that among such subfields, there is a smallest one C_{0}, i.e., C_{0} is contained in any other C.*
2. *What is the degree of F/C_{0} ?*
3. *(Optional question), what is C_{0}?*

题 10. *Let F be field with characteristic not equal to 2. Let $\alpha \in F^{\times} \backslash\left(F^{\times}\right)^{2}, a, b \in F$. Let $K=F[\sqrt{\alpha}]$ be a quadratic extension of F and $L=K[\sqrt{a+b \sqrt{\alpha}}]$ a quadratic extension of K. Prove that L/F is Galois with cyclic Galois group if and only if $a^{2}-b^{2} \alpha=c^{2} \alpha$ for some $c \in F^{\times}$.*