代数 2 H 班 作业 5

2023 年 8 月 2 日

题 1. *Let p, q be prime numbers. Let G be a group of order pq. Prove that G is solvable.*

题 2. *Find the minimal number n such that there exists a solvable group of order n.*

题 3. *Let $f \in F[x]$ be a degree-n polynomial with roots $\alpha_1, \cdots, \alpha_n$. Define the discriminant of f by $D(f) = \Pi_{i<j}(\alpha_i - \alpha_j)^2$. The Galois group G_f acts on the roots generates an injective group homomorphism $\rho \colon G_f \to S_n$. Prove that the image of ρ is in A_n if and only if $D(f)$ is a square in F. Compute $D(f)$ in terms of the coefficients of f when $\deg f = 2$ and $\deg f = 3$.*

题 4. *Find a degree-3 Galois extension of $\mathbb{Q}$. (分类这种 extension, 供思考, 不用交)*

题 5. *For a degree-three separable irreducible polynomial $f(x) \in F[x]$. Discuss how to determine the Galois group of G_f over F.*

题 6 (Milne)**.** *Let $f(X)$ be an irreducible polynomial in $\mathbb{Q}[X]$ with both real and nonreal roots. Show that its Galois group is nonabelian. Can the condition that f is irreducible be dropped?*

题 7 (Milne)**.** *Let $f(X) = X^5 + aX + b, a, b \in \mathbb{Q}$. Show that $G_f \approx D_5$ (dihedral group) if and only if*

1. *$f(X)$ is irreducible in $\mathbb{Q}[X]$, and*
2. *the discriminant $D(f) = 4^4a^5 + 5^5b^4$ of $f(X)$ is a square, and*

3. *the equation $f(X) = 0$ is solvable by radicals.*

题 8 (Milne)**.** *Give an example of a field extension E/F of degree 4 such that there does not exist a field M with $F \subset M \subset E, [M:F] = 2$*

题 9 (Milne)**.** *Let F be a Galois extension of $\mathbb{Q}$, and let α be an element of F such that $\alpha F^{\times 2}$ is not fixed by the action of $\mathrm{Gal}(F/\mathbb{Q})$ on $F^{\times}/F^{\times 2}$. Let $\alpha = \alpha_1, \ldots, \alpha_n$ be the orbit of α under $\mathrm{Gal}(F/\mathbb{Q})$. Show:*

1. *$F\left[\sqrt{\alpha_1}, \ldots, \sqrt{\alpha_n}\right]/F$ is Galois with commutative Galois group contained in $(\mathbb{Z}/2\mathbb{Z})^n$.*

2. *$F\left[\sqrt{\alpha_1}, \ldots, \sqrt{\alpha_n}\right]/\mathbb{Q}$ is Galois with noncommutative Galois group contained in $(\mathbb{Z}/2\mathbb{Z})^n \rtimes \mathrm{Gal}(F/\mathbb{Q})$.*

题 10. *Let n be an integer. Assume F contains n-th root of unity. Let $a \in F$. For each prime $p \mid n$, the element $a \in F$ has no pth root in F. (Or $a \notin F^p$). Prove that $x^n - a$ is irreducible in $F[x]$. (Similar result still holds if F does not contain n-th root of unity with additional condition that if $4 \mid n$, then $a \notin -4F^4$. See Lang Theorem VI 9.1.)*

题 11. *Describe the elements in the Galois group of splitting field of $x^8 - 2$ over $\mathbb{Q}$. Find all the intermediate extensions.*

题 12. *Find the centralizer of complex conjugation in $Aut_{\mathbb{Q}}(\mathbb{C})$.*