代数 2 H 班 作业 6

2023 年 8 月 2 日

题 1. *Let* $f(x) = (x_1)^4 + (x_2)^4 + (x_3)^4$*. Write* f *as a polynomial of elementary polynomials.*

题 2. *Let* F *be field with characteristic zero. Let* $p_k = (x_1)^k + \cdots + (x_n)^k$*. Prove the ring homomorphism*

$$F[y_1, \cdots, y_n] \to F[x_1, \cdots, x_n]^{S_n}$$

defined by $g(y_1 \cdots y_n) \mapsto g(p_1 \cdots p_n)$ *is a ring isomorphism. If* F *is replaced by a commutative ring* R*, does the same conclusion still hold? (Optional, what additional conditions do you need?)*

题 3. *Let* F *be a field. Consider the Galois extension* $F(x_1, x_2, x_3)/F(s_1, s_2, s_3)$ *where* s_1, s_2, s_3 *are elementary symmetric polynomials. Find all the intermediate extensions and draw the Galois correspondence by an extension diagram. Please indicate which ones are conjugate to each other. (Optional: draw similar diagram for four variables.)*

题 4 (Solve cubic equation)**.** *Let* F *be a field with third root of unity and* $\mathrm{char}(F) = 0$*. Solve the cubic equation* $x^3 + px + q$ *with radicals over* $F(p, q)$ *explicitly.*

题 5. *Find all the subgroups of* S_4 *acting transitively on* $\{1, 2, 3, 4\}$*.*

题 6. *Let* E *be the splitting field of an irreducible separable polynomial* $f \in F[x]$*. If no root of* f *generates* E*, show that* $Gal(E/F)$ *contains a nonnormal subgroup.*

题 7. *Let p be a prime number. Let H be a subgroup of order p in S_p. Find the normalizer of H in S_p.*

题 8.

1. *Let G be a finite group acting transitively on set X. Let H be a normal subgroup of G. Prove that every H orbit in X has the same length.*

2. *Let p be a prime number. Let G be a solvable subgroup of S_p such that $p \mid |G|$. Prove that the Sylow p subgroup of G is normal in G.*

题 9 (Galois' theorem)**.** *Let F be a field and $\mathrm{char}(F) = 0$. Let $f \in F[x]$ be an irreducible polynomial and $\deg f = p$ is a prime number. Denote by K the splitting field of F. Prove that f is solvable with radicals if and only if for any two distinct roots α_i, α_j of f, we have $K = F[\alpha_i, \alpha_j]$.*

题 10. *Use the previous theorem to show that the example by Brauer in class is not solvable by radicals. Or in other words, a degree-p irreducible polynomial over $\mathbb{Q}$ with exactly $p-2$ real roots is not solvable by radicals.*