代数 2 H 班 作业 7

2023 年 8 月 2 日

题 1 (Milne)**.** *Let H be a transitive subgroup of S_n containing a transposition and $(n-1)$-cycle. Prove that $H = S_n$.*

题 2 (Milne)**.** *Select separable monic polynomials of degree n, f_1, f_2, f_3 with coefficients in $\mathbb{Z}$ with the following factorizations:*

1. *f_1 is irreducible* mod2 *;*
2. *f_2 = (degree 1) (irreducible of degree $n-1$)* mod3*;*
3. *f_3 = (irreducible of degree 2)(product of 1 or 2 irreducible polynomials of odd degree)* mod5*.*

Take

$$f = -15f_1 + 10f_2 + 6f_3.$$

Prove that the Galois group of f over $\mathbb{Q}$ is S_n.

题 3. *Prove that every finite abelian group can be realized as the Galois group of $K/\mathbb{Q}$.*

题 4 (Lang)**.** *Prove that there are infinitely many non-zero relatively prime integers a, b such that $-4a^3 - 27b^2$ is a square in $\mathbb{Z}$.*

题 5. *Let K be a finite extension of $\mathbb{Q}$. Prove that there are only finitely many roots of unity in K.*

题 6 (Lang)**.** *What is the Galois group over the rationals of the following polynomials:*

1. *$X^4 + 2X^2 + X + 3$*

2. $X^4 + 3X^3 - 3X - 2$

3. $X^6 + 22X^5 - 9X^4 + 12X^3 - 37X^2 - 29X - 15$

[Hint: Reduce $\mathrm{mod} 2, 3, 5$*.]*

題 7. *Please find an example such that* $\Phi_d(x)$ *is reducible when modulo some prime number* $p \not| d$*.*

題 8. *Let* α *be an algebraic integer and* $f(x)$ *its minimal polynomial over* $\mathbb{Q}$*. Assume all the roots of* $f(x)$ *in* $\mathbb{C}$ *have absolute value* 1*. Prove that* α *is a root of unity.[Hint: consider all the powers of* α *and prove that the coefficients of these minimal polynomials are bounded.]*

題 9. *Let* F *be a field and* $a_1, \cdots, a_n \neq 0 \in F$ *be* n *different element. Prove that there exists* $k \in Z$ *such that* $a_1^k + \cdots + a_n^k \neq 0$*. Can you require that* $k \geq 0$*?*

題 10 (Milne)**.** *Let* E *be a finite separable extension of* F *of degree* m*. Let* $\alpha_1, \ldots, \alpha_m$ *be a basis for* E *as an* F*-vector space, and let* $\sigma_1, \ldots, \sigma_m$ *be distinct* F*-homomorphisms from* E *into a field* Ω*. Then the matrix whose* (i, j) *th-entry is* $\sigma_i \alpha_j$ *is invertible.*