代数 2 H 班 作业 8

2023 年 8 月 2 日

题 1. *Let A, B be integers such that $A^2 - 4B$ is not a square. Parametrize the integer solutions to $x^2 + Axy + By^2 = z^2$ similar as $x^2 + y^2 = z^2$.*

题 2. *Find the primitive element for all the intermediate extensions of $\mathbb{Q}[\zeta_{17}]/\mathbb{Q}$.*

题 3. *Let R be a noetherian ring. Prove that $R[[x]]$ is noetherian.*

题 4 (Eisenbud)**.** *Let $R = R_0 \oplus R_1 \oplus \cdots$ be a graded ring. Prove that the following are equivalent:*

1. *R is Noetherian.*
2. *R_0 is Noetherian and the irrelevant ideal $R_1 \oplus R_2 \oplus \cdots$ is finitely generated.*
3. *R_0 is Noetherian and R is a finitely generated R_0-algebra.*

题 5 (Eisenbud)**.** *Although the Noetherian property does not usually pass from a ring to a subring, it does when the subring is a summand:*

Let $R \subset S$ be rings, and assume that R is a summand of S as an R module, that is, there is a homomorphism $\varphi : S \to R$ of R-modules fixing every element of R. Prove that if S is Noetherian, then R is Noetherian.

题 6 (Eisenbud)**.** *Let k be a field. Compute the Hilbert function and polynomial for the ring*

$$k[x, y, z, w]/(x, y) \cap (z, w)$$

corresponding to the disjoint union of two lines in projective 3 -space. Compare these to the Hilbert function and polynomial of the ring corresponding to one projective line, $k[x, y]$.

题 7 (Eisenbud)**.** *Let k be a field. Let $I \subset k[x,y,z,w]$ be the ideal generated by the 2×2 minors of the matrix*

$$\begin{pmatrix} x & y & z \\ y & z & w \end{pmatrix}$$

that is, $I = (yw - z^2, xw - yz, xz - y^2)$. Show that $R = k[x,y,z,w]/I$ is a finitely generated free module over $S = k[x,w]$. Exhibit a basis for R as an S-module. Show that there is a ring homomorphism $R \to k[s,t]$ such that $x \mapsto s^3, y \mapsto s^2t, z \mapsto st^2, w \mapsto t^3$. Use the basis you constructed to show that it is a monomorphism. Conclude that I is prime. From the rank of R as a free S-module, and the degrees of the generators, deduce the Hilbert function of R. Show that R is not finitely generated as a module over $k[x,y]$.

题 8 (Example by a student，please contact me with your name, I forgot to ask)**.** *Let k be a field. $S \subset k[x,y]$ be a subalgebra generated by $x^i y^j$ such that $1 \leq i < j$. Prove that S is not finitely generated k-algebra.*