代数 2 H 班 作业 9

2023 年 8 月 2 日

We always denote by R a commutative ring.

题 1.

定义 1. *If $f \in R$ satisfies $f^n = 0$ for some positive integer, we call f a nilpotent element. The set of nilpotent elements of R is called the radical or nilradical of R and denoted by $r((0))$. More generally, we define $r(I) = \{f \in R \mid \text{there exists positive integer } n, \text{such that } f^n \in I\}$*

Prove the following facts about radical.

1. *$r(I)$ is an ideal of R.*
2. *$r(I)$ is the intersection of all prime ideals of R containing I.*
3. *We call an ideal radical ideal if $r(I) = I$. Prove that there is a one-to-one correspondence between the set of radical ideals and irreducible closed subets of $\operatorname{Spec}(R)$ by $I \mapsto Z(I)$. And this map reverses the inclusion relation.*

题 2 (Atiyah & Macdonald)**.** *Prove the following facts about radicals.*

1. $r(\mathfrak{a}) \supseteq \mathfrak{a}$
2. $r(r(\mathfrak{a})) = r(\mathfrak{a})$
3. $r(\mathfrak{a}\mathfrak{b}) = r(\mathfrak{a} \cap \mathfrak{b}) = r(\mathfrak{a}) \cap r(\mathfrak{b})$
4. $r(\mathfrak{a}) = (1) \Leftrightarrow \mathfrak{a} = (1)$
5. $r(\mathfrak{a} + \mathfrak{b}) = r(r(\mathfrak{a}) + r(\mathfrak{b}))$

6. *if $\mathfrak{p}$ is prime, $r(\mathfrak{p}^n) = \mathfrak{p}$ for all $n > 0$.*

题 3 (Atiyah & Macdonald)**.** *The Jacobson radical ideal I is defined to be the intersection of all the maximal ideals of R. It can be characterized as follows:*

$x \in I \Leftrightarrow 1 - xy$ *is a unit in R for all $y \in R$.*

题 4 (Atiyah & Macdonald)**.** *Let x be a nilpotent element of a ring A. Show that $1 + x$ is a unit of A. Deduce that the sum of a nilpotent element and a unit is a unit.*

题 5 (Atiyah & Macdonald)**.** *Let A be a ring and let $A[x]$ be the ring of polynomials in an indeterminate x, with coefficients in A. Let $f = a_0 + a_1x + \cdots + a_nx^n \in A[x]$. Prove that*

1. *f is a unit in $A[x] \Leftrightarrow a_0$ is a unit in A and $a_1, \ldots, a_n$ are nilpotent. [If $b_0 + b_1x + \cdots + b_mx^m$ is the inverse of f, prove by induction on r that $a_n^{r+1}b_{m-r} = 0$. Hence show that a_n is nilpotent, and then use previous excercise.]*

2. *f is nilpotent $\Leftrightarrow a_0, a_1, \ldots, a_n$ are nilpotent.*

3. *f is a zero-divisor $\Leftrightarrow$ there exists $a \neq 0$ in A such that $af = 0$. [Choose a polynomial $g = b_0 + b_1x + \cdots + b_mx^m$ of least degree m such that $fg = 0$. Then $a_nb_m = 0$, hence $a_ng = 0$ (because a_ng annihilates f and has degree $< m$). Now show by induction that $a_{n-r}g = 0 (0 \leqslant r \leqslant n)$.]*

4. *f is said to be primitive if $(a_0, a_1, \ldots, a_n) = (1)$. Prove that if $f, g \in A[x]$, then fg is primitive $\Leftrightarrow f$ and g are primitive.*

题 6 (Atiyah & Macdonald)**.** *In the ring $A[x]$, the Jacobson radical is equal to the nilradical.*

Later we will see that these two are equal for finitely generated algebra R over a field (Hilbert's Nullstellensatz). This actually means that $\mathrm{Spec}_m(R)$ can "recover" $\mathrm{Spec}(R)$. Think about why.

题 7 (参考 Atiyah & Macdonald 第一章 17 题, 选做)**.** *Prove that* $\mathrm{Spec}(R)$ *is compact under Zariski topology.*

题 8. *Let* $X = \mathrm{Spec}(R)$ *and* $f \in R$. *Denote by* $U_f = X - Z(f)$. *Let* $S = R[x]/(xf-1)$. *Prove that* $\mathrm{Spec}(S)$ *is homeomorphic to* U_f *induced by the natural ring homomorphism* $R \to S$.

题 9. *Prove the spectrum of product ring* $R \times S$ *is homeomorphic to the disjoint union* $\mathrm{Spec}(R)$ *and* $\mathrm{Spec}(S)$.

题 10.

定义 2. *A topological space* X *is called Noetherian if it satisfies the descending chain condition for closed subsets: for any sequence*

$$Y_1 \supseteq Y_2 \supseteq \cdots$$

of closed subsets Y_i *of* X, *there is an integer* m *such that* $Y_m = Y_{m+1} = \cdots$.

Prove the following

1. *The definition is equivalent to every collection of closed subsets of* X *has a minimal element under inclusion.*
2. *The definition is equivalent to every open subset of* X *is compact.[选做]*
3. *Every closed subset of Noetherian space* X *is a finite union of irreducible subsets.*
4. *If* R *is Noetherian, then* $\mathrm{Spec}(R)$ *is Noetherian.*

题 11. *Describe points and closed subsets of* $\mathrm{Spec}(\mathbb{C}[x,y]/(x^2+y^2))$ *and* $\mathrm{Spec}(\mathbb{R}[x,y]/(x^2+y^2))$.