Algebraic curves

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Course website:
chenglongyu.github.io/Algebraiccurves
Fall 2023

Motivation

1. Meromorphic functions

$$\begin{tikzcd} U_t \arrow[d, "f"] \\ \mathbb{C} \cup \{\infty\} \end{tikzcd}$$

U_t region in $\mathbb{C}$

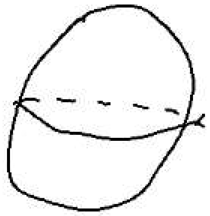
$z(t)$

we form a local coordinate around z by

$w = \frac{1}{z}$. $w(t)$ order of zero = order of pole for z

Or more precisely, we have the construction for Riemann sphere

$$\mathbb{P}^1 = \underset{z}{\mathbb{C}} \underset{\mathbb{C}^\times}{\cup} \underset{w}{\mathbb{C}} \qquad w = \frac{1}{z}$$



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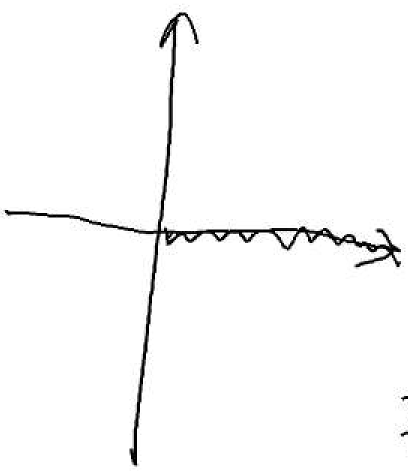
2. Multivaluedness of holomorphic functions

2.1 $f(z) = \sqrt{z}$



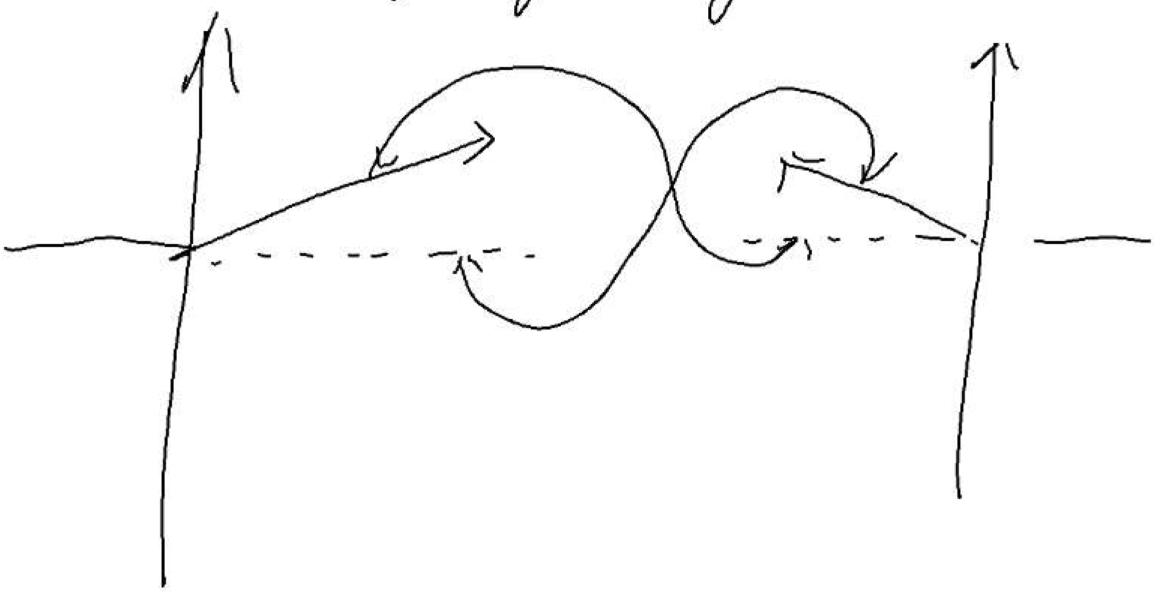
$\mathbb{C} \setminus (0, \infty)$

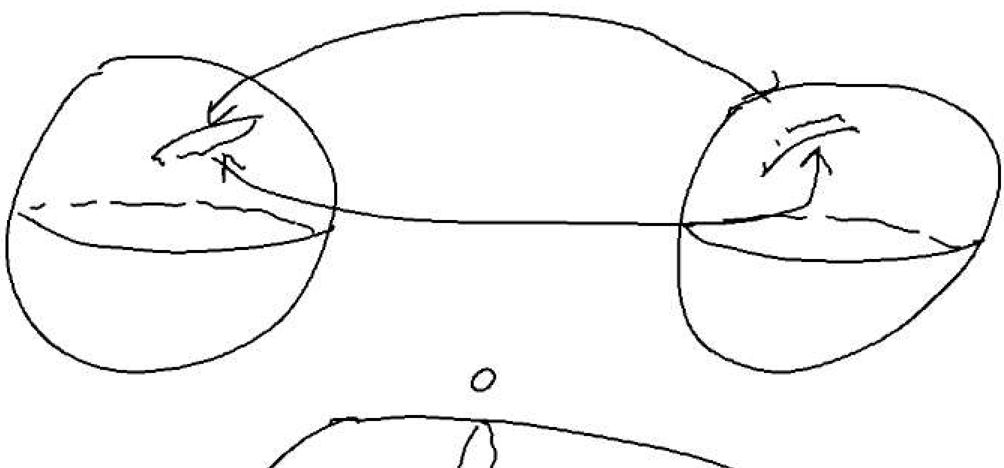
we can define $\sqrt{z}$ by

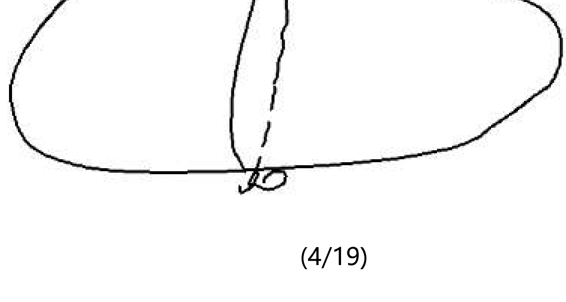
$z = re^{i\theta}$ $r \geq 0$

$0 < \theta < 2\pi$. $\sqrt{z} = \sqrt{r}\, e^{i\frac{\theta}{2}}$

Look at how $\sqrt{z}$ "jumps" between the two sides of $(0, \infty)$, we make the following gluing



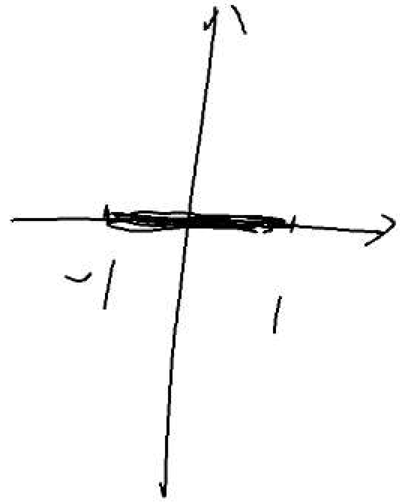




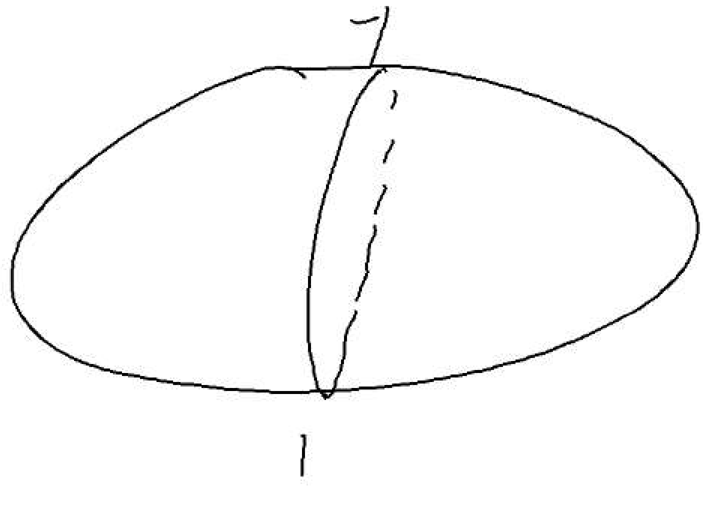
2.2 Similarly for

$$f(z) = \sqrt{1-z^2}$$

① | $(-1, 1)$



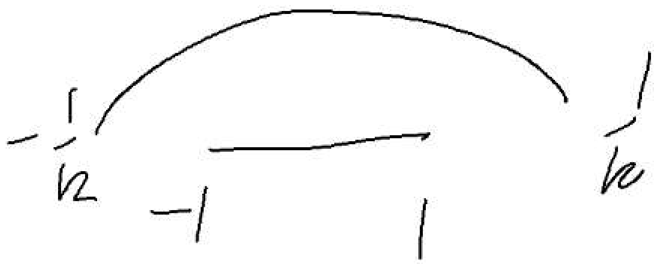
glue two copies together we obtain

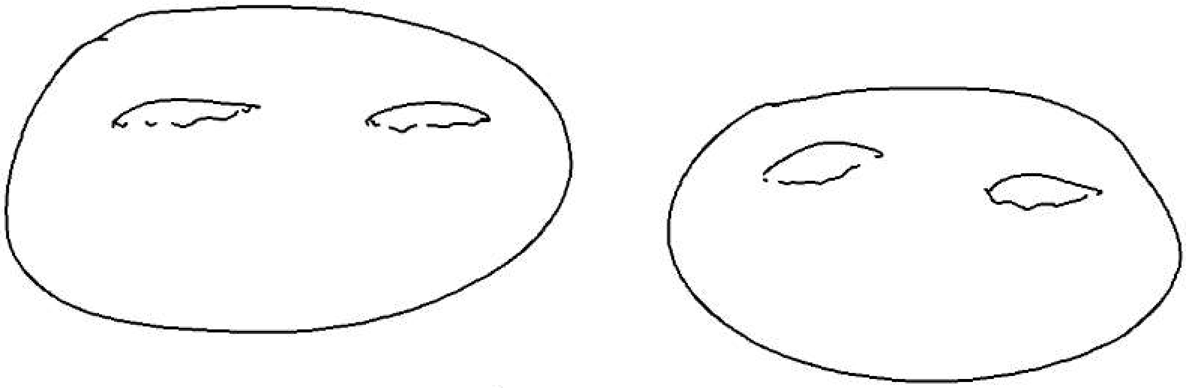


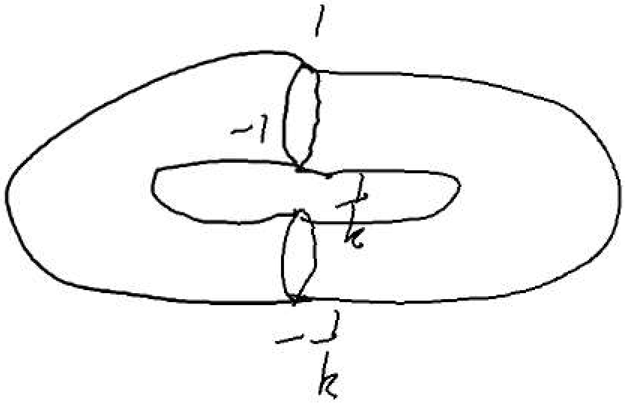
a sphere

2.3

$$f(z) = \sqrt{(1-z^2)(1-k^2z^2)} \qquad k \neq \pm 1$$







torus

Summarize a little bit

$$y=\sqrt{z}, \qquad y^2=z$$

2.1 $\{(y,z)\subset \mathbb{C}^2 \mid y^2=z\}\cup\{\infty\}$

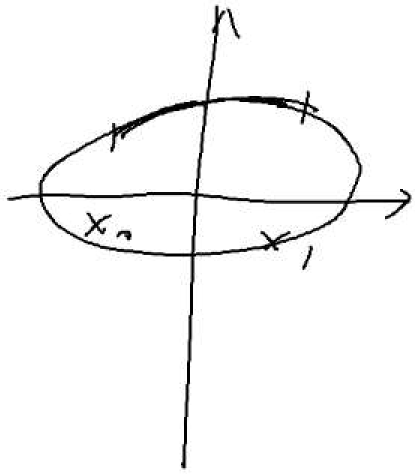
$$\begin{array}{ccc} \downarrow & & \downarrow \\ y=\sqrt{z} & & z \end{array}$$

2.2 $\{(y,z)\subset \mathbb{C}^2 \mid y^2=1-z^2\}\cup\{\infty\}$

2.3 $\{(y,z)\subset \mathbb{C}^2 \mid y^2=(1-z^2)(1-k^2z^2)\}$
$\cup\{\infty_1,\infty_2\}$

3. Abelian integrals

3.1 Arc length of ellipse



$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

$$b < a$$

$$x = a\cos\theta, \quad y = b\sin\theta$$

$$\int_{\theta_0}^{\theta_1} \sqrt{a^2\sin^2\theta + b^2\cos^2\theta}\, d\theta$$

$$= a\int_{\theta_0}^{\theta_1} \sqrt{1 - k^2\sin^2\theta}\, d\theta$$

$$k = \sqrt{\frac{a^2 - b^2}{a^2}}$$

$$\int_{\theta_0}^{\theta_1} \sqrt{1-k^2\sin^2\theta}\; d\theta \,, \qquad \sin\theta = z$$

$$dz = \cos\theta\, d\theta$$

$$= \int_{z_0}^{z_1} \frac{\sqrt{1-k^2z^2}}{\sqrt{1-z^2}}\, dz \quad \text{(elliptic}$$

$$= \int_{z_0}^{z_1} \frac{1-k^2z^2}{\sqrt{(1-k^2z^2)(1-z^2)}}\, dz \quad \text{integral of 2nd kind)}$$

when $k=0$, circle

$$\Rightarrow \int_{z_0}^{z_1} \frac{1}{\sqrt{1-z^2}}\, dz = \arcsin z_1 - \arcsin z_0$$

$$\text{or} \int_0^{z} \frac{1}{\sqrt{1-t^2}}\, dt = \theta$$

$$z = \sin\theta$$

Classical theory of "addition formula"

$$\sin(\alpha+\beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta$$

$$= \sin\alpha\sqrt{1-\sin^2\beta} + \sqrt{1-\sin^2\alpha}\,\sin\beta$$

In terms of integration

$$\int_0^{z_1} \frac{1}{\sqrt{1-t^2}}\,dt + \int_0^{z_2} \frac{1}{\sqrt{1-t^2}}\,dt$$

$$= \int_0^{z_1\sqrt{1-z_2^2}+\sqrt{1-z_1^2}\,z_1} \frac{1}{\sqrt{1-t^2}}\,dt$$

For an analogue, $\sin t = z$

(elliptic integral of 1st kind)

$$\alpha = \int_0^{\arcsin z} \frac{1}{\sqrt{1-k^2\sin^2 t}}\, dt = \mathrm{sn}^{-1}(z)$$

$$\mathrm{sn}(\alpha+\beta) = \frac{\mathrm{sn}(\alpha)\sqrt{(1-\mathrm{sn}^2\beta)(1-k^2\mathrm{sn}^2\beta)} + \mathrm{sn}(\beta)\cdot\sqrt{(1-\mathrm{sn}^2\alpha)(1-k^2\mathrm{sn}^2\alpha)}}{1-k^2\mathrm{sn}^2\alpha\,\mathrm{sn}^2\beta}$$

sn cannot be expressed as an elementary function

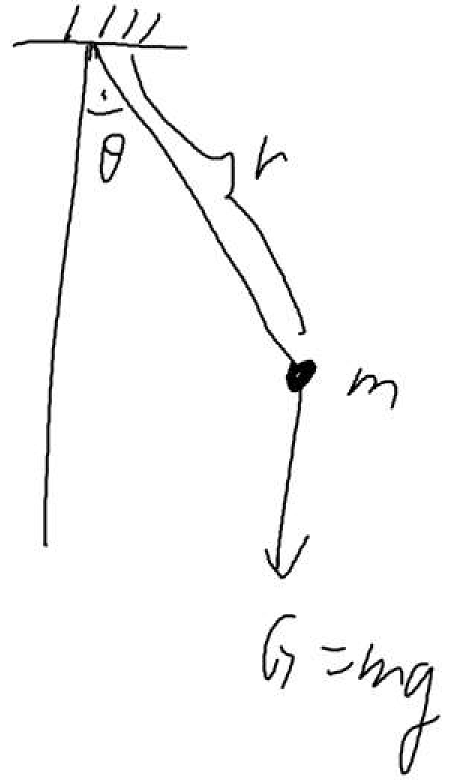
related to the fact that

$y^2 = 1-z^2$ is a Riemann sphere

$y^2 = (1-z^2)(1-k^2z^2)$ is not.

Abelian integral appears very naturally, another example

3.2 Simple pendulum.



m is released at $\theta = \alpha$ with zero initial velocity.

Conservation of energy $\Rightarrow$

$$\frac{1}{2} m r^2 \left(\frac{d\theta}{dt}\right)^2 = mgr\cos\theta - mgr\cos\alpha$$

$$\left(\frac{d\theta}{dt}\right)^2 = 2\frac{g}{r}(\cos\theta - \cos\alpha)$$

$$= 4\frac{g}{r}\left(\sin^2\frac{\alpha}{2} - \sin^2\frac{\theta}{2}\right)$$

An appoximation with θ very small

$\sin\frac{\theta}{2} \doteq \frac{\theta}{2} \Rightarrow$

$$\frac{d\theta}{dt} = \sqrt{\frac{g}{r}(\alpha^2 - \theta^2)}$$

$$t = \int_0^\theta \sqrt{\frac{r}{g}} \frac{1}{\sqrt{\alpha^2 - s^2}}\, ds$$

$$\text{Period} = 4 \cdot \int_0^\alpha \sqrt{\frac{r}{g}} \frac{1}{\sqrt{\alpha^2 - s^2}}\, ds$$

$$= 4\int_0^1 \sqrt{\frac{r}{g}} \frac{1}{\sqrt{1-x^2}}\, dx = 2\pi\sqrt{\frac{r}{g}}$$

Without approximation

$$\left(\frac{d\theta}{dt}\right)^2 = 4\frac{g}{r}\left(\sin^2\frac{\alpha}{2} - \sin^2\frac{\theta}{2}\right)$$

substitution $$\frac{\sin\frac{\theta}{2}}{\sin\frac{\alpha}{2}} = \sin\varphi$$

$$\Rightarrow \quad \frac{\frac{1}{2}\cos\frac{\theta}{2}\frac{d\theta}{dt}}{\sin\frac{\alpha}{2}} = \cos\varphi\frac{d\varphi}{dt}$$

$$\left(\frac{d\varphi}{dt}\right)^2 = \frac{1}{\cos^2\varphi}\cdot\frac{1}{4}\frac{\cos^2\frac{\theta}{2}}{\sin^2\frac{\alpha}{2}}\cdot\left(\frac{d\theta}{dt}\right)^2$$

$$= \frac{1}{\cos^2\varphi}\,\frac{1}{4}\,\frac{1-\sin^2\varphi\sin^2\frac{\alpha}{2}}{\sin^2\frac{\alpha}{2}}$$

$$4\frac{g}{r}\cdot\left(\sin^2\frac{\alpha}{2} - \sin^2\frac{\alpha}{2}\sin^2\varphi\right)$$

$$= \frac{g}{r}\left(1 - \sin^2\frac{\alpha}{2}\sin^2\varphi\right)$$

$$k = \sin\frac{\alpha}{2}$$

$$t = \sqrt{\frac{r}{g}} \cdot \int_0^{\varphi} \frac{ds}{\sqrt{1-k^2\sin^2 s}}$$

elliptic integral

$$\Rightarrow \varphi = sn\left(\sqrt{\frac{g}{r}}\, t\right)$$

Period $\quad T = 4\sqrt{\frac{r}{g}} \int_0^{\frac{\pi}{2}} \frac{ds}{\sqrt{1-k^2\sin^2 s}}$

$$= 2\pi\sqrt{\frac{r}{g}} \cdot \left(1 + \left(\frac{1}{2}\right)^2 k^2 + \left(\frac{1\cdot 3}{2\cdot 4}\right)^2 k^4 + \cdots\right)$$

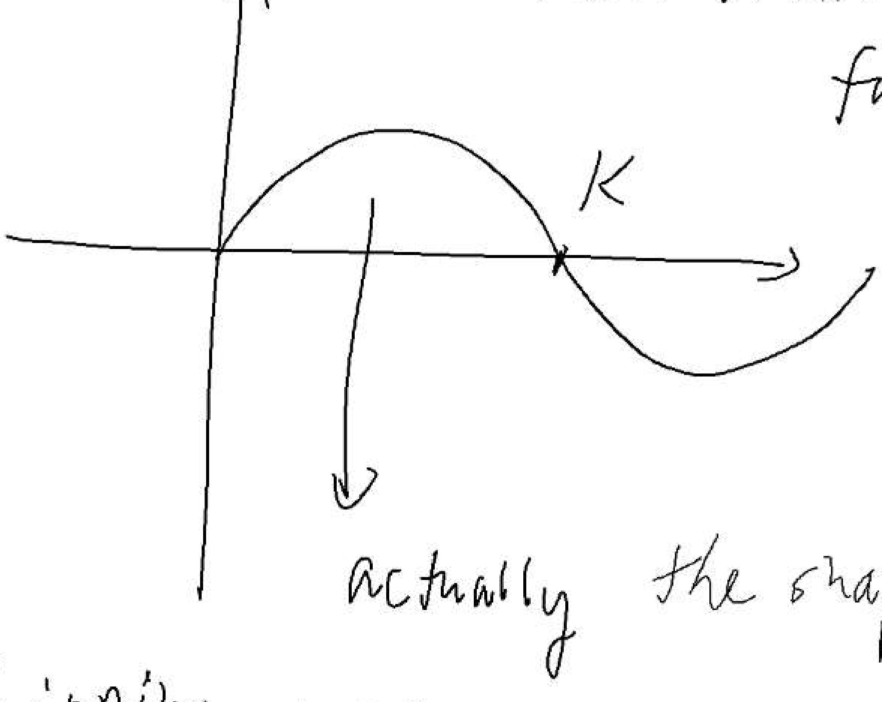
Not related to α

elliptic functions

$sn(k, u)$ appears

naturally in physics, analysis.

shape of the function as real valued function



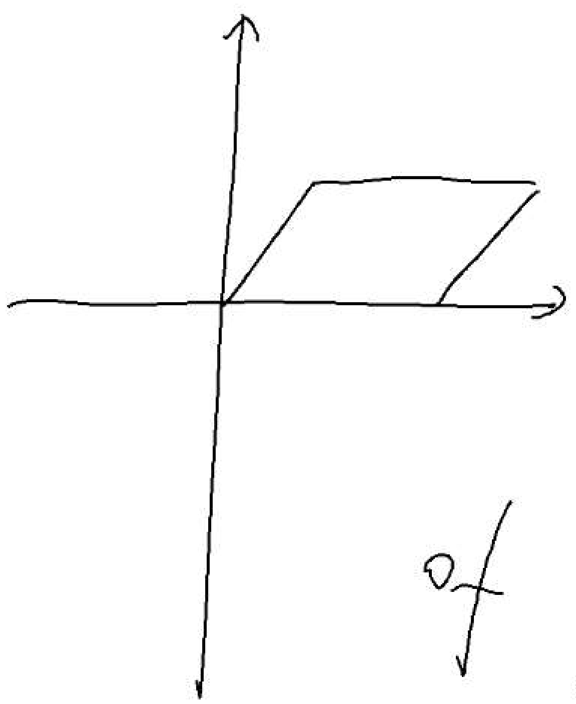
actually the shape of skipping rope

More interestingly,

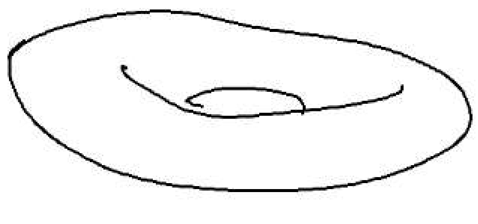
$sn(k,u)$ can be extended

as double periodic meromorphic

function



This is related to the topology of $y^2=(1-x^2)(1-k^2x^2)$



In general, $P(x,y)=0$. $y=f(x)$

$$\int R(x, f(x))\, dx$$

$\deg P = 0, 1, 2$, can be expressed as elementary functions

$\deg P \geq 3$ in general no

related to the topology of P

$\{P(x,y)=0\}$ algebraic curves.

Arithmetics

$x^n+y^n=z^n$, ① $n=2$ $x^2+y^2=1$

$$x=\frac{2t}{1+t^2}, \quad y=\frac{1-t^2}{1+t^2}$$

$t \in \mathbb{Q}$, rational solutions to

$x^2+y^2=1$

② $n \geq 3$. no nontrivial solution

③ More generally

$p(x,y)=0$

deg p is related to the number of solutions
