Riemann surface and complex structure

Ex: $S^2 = \underset{z}{\mathbb{C}} \underset{\mathbb{C}^\times}{\cup} \underset{w}{\mathbb{C}} \quad z = \frac{1}{w}$

X topological space

① Hausdorff

② C_2 countable basis

Defn (Partition of unity)

$\exists\ \varphi_n : X \to \mathbb{R}$ continuous with cpct support.

$$\sum_{n=1}^{\infty} \varphi_n = 1$$

Defn: (cplx charts)

① $\{U_i\}_{i\in I}$ open covering of X

② $f_i : U_i \to \mathbb{C}$

f_i homeomorphism onto open subset $f_i(U_i) \subset \mathbb{C}$

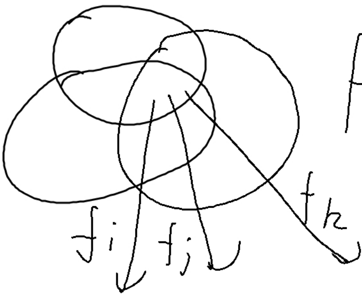
③

biholomorphic

$f_j f_i^{-1} : f_i(U_i \cap U_j) \to f_j(U_i \cap U_j)$

$f_{ji} = f_j f_i^{-1}$ are called transition functions

check:

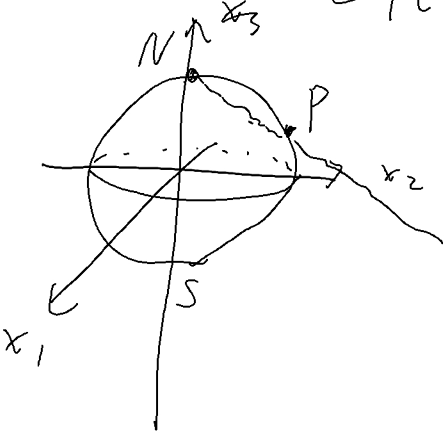


$f_{ik} f_{kj} f_{ji} = id$

Defn: X with cplx charts is called a Riemann surface.

Ex: Riemann sphere S^2

$= \{(x_1, x_2, x_3) \in \mathbb{R}^3 \mid x_1^2 + x_2^2 + x_3^2 = 1\}$



$N = (0,0,1)$

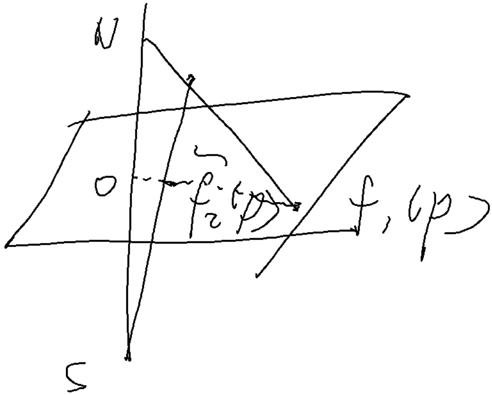
$S = (0,0,-1)$

$U_1 = S^2 \setminus \{N\}$, f_1: projection from N

find the coordinates

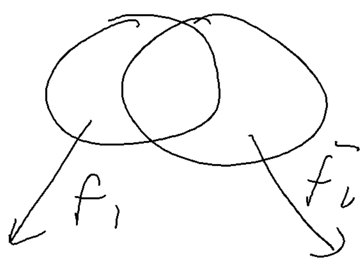
$$(x_1, x_2, x_3) \mapsto \left(\frac{x_1}{1-x_3}, \frac{x_2}{1-x_3}\right)$$

$U_2 = S^2 \setminus \{S\}$. $\tilde{f}_2$: Projection from S.



$$\triangle N O f_1(p) \sim \triangle S O \tilde{f}_2(p)$$

$$\Rightarrow |O\tilde{f}_2(p)| \cdot |O f_1(p)| = 1$$



$$\mathbb{C} \setminus \{0\} \longrightarrow \mathbb{C} \setminus \{0\}$$

$$z \longmapsto z \cdot \frac{1}{|z|^2} = \overline{\left(\frac{1}{z}\right)}$$

$$c : \mathbb{C} \to \mathbb{C}$$
$$w \mapsto \overline{w}$$

$$f_1 = c \circ \widetilde{f_2}$$

$$\Rightarrow \quad f_2 f_1^{-1} : \mathbb{C}^\times \to \mathbb{C}^\times$$
$$z \mapsto \frac{1}{z}$$

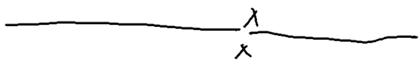
Or: $S^2 = \underset{z}{\mathbb{C}} \sqcup \underset{w}{\mathbb{C}} \Big/ \sim$

$\sim$: $z \sim w$ iff $z = \frac{1}{w} \neq 0$

Note: Hausdorffness (seperatedness) is not automatic,

$\mathbb{C} \sqcup \mathbb{C}$ $\qquad z = w$

$z \in \mathbb{C}^\times \ni w$



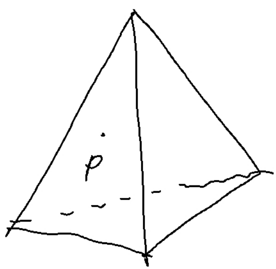
$z=0$, $w=0$ cannot be seperated by open nbhd

Defn: (equivalent charts) $\{U_i\}_{i\in I}$, $\{V_j\}_{j\in J}$ cplx charts on X

$\{U_i, V_j\}_{i\in I, j\in J}$ cplx charts, "equivalent" equivalence classes of charts are called "cplx structures"

or we can take the "maximal" charts. (Riemann surface)

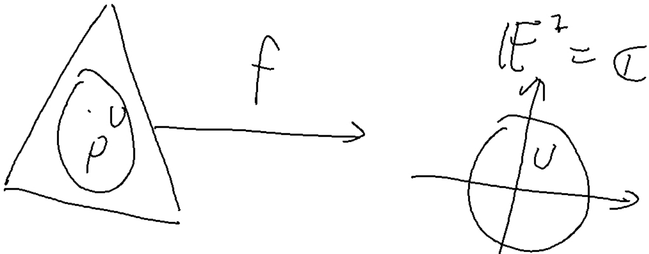
More examples:



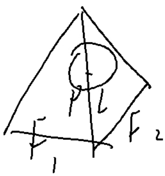
Convex polyhedra in Euclidean 3D space $\mathbb{E}^3$ (topologically sphere)

p is the interior of the face F

F can be isometrically embedded into $\mathbb{E}^2$

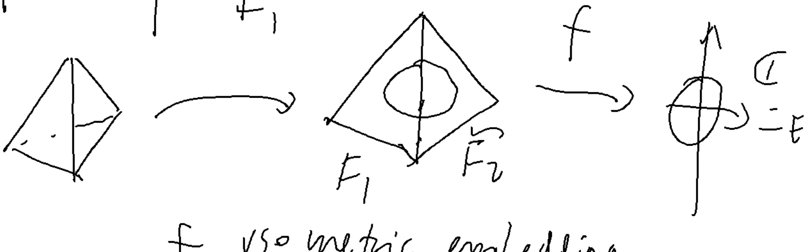


choose f preserving orientation. isometric embedding

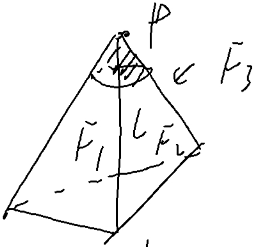


If $p \in l = F_1 \cap F_2$

Flatten F_1, F_2, i.e. rotate F_2 along l to the plane of F_1

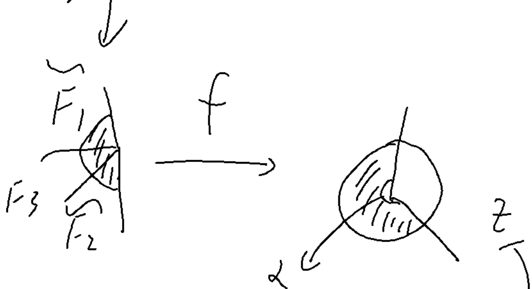


f isometric embedding



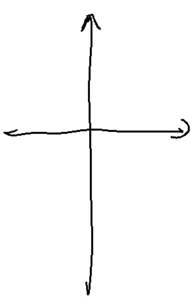
p vertex

cut along l, rotate F_1, F_2 to the plane of F_3



$0 < \alpha < 2\pi$

f isometric embedding

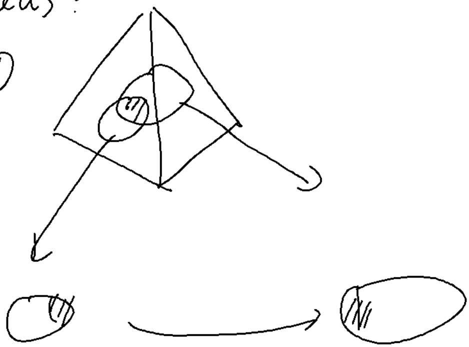


$z^{\frac{2\pi}{\alpha}}$

Prove this is a "cplx chart"

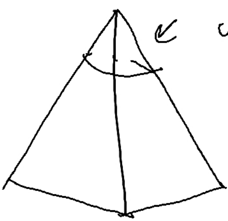
Ideas:

①



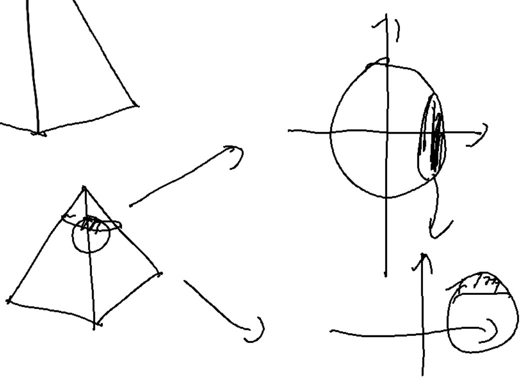
rotation + parallel transport

②



← different cut ⇒ rotation

③



$$z \downarrow z^{\frac{2}{2\pi}}$$

Torus: $w_1, w_2 \in \mathbb{C}$,

w_1, w_2 $\mathbb{R}$-linearly independent

$\Rightarrow \mathbb{Z}w_1 + \mathbb{Z}w_2$ discrete in $\mathbb{C}$

(proving this is an excercise)

Quotient group $\mathbb{C}/\mathbb{Z}w_1+\mathbb{Z}w_2$ with quotient topology (A subset is open iff its preimage in $\mathbb{C}$ is open)

$\mathbb{C}/\mathbb{Z}w_1+\mathbb{Z}w_2$ Hausdorff

$(U+\mathbb{Z}w_1+\mathbb{Z}w_2)\cap V = \phi$

$(U-V)\cap \mathbb{Z}w_1+\mathbb{Z}w_2 = \phi$

$p_1 + W = U$. $\quad W \ni 0$ nbhd

$p_2 + W = V$

$U - V = (p_1 - p_2) + (W - W)$

$p_1 - p_2 \notin \mathbb{Z}w_1 + \mathbb{Z}w_2$. choose

$\widetilde{W}$ s.t. $(p_1 - p_2) + \widetilde{W} \cap \mathbb{Z}w_1 + \mathbb{Z}w_2 = \phi$

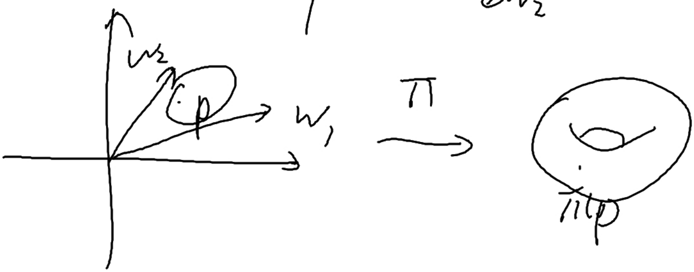
choose W s.t. $W - W \subset \widetilde{W}$

$\Rightarrow \pi(U) \cap \pi(V) = \phi$ in

$\mathbb{C} \xrightarrow{\pi} \mathbb{C}/\mathbb{Z}w_1 + \mathbb{Z}w_2$

$\mathbb{C}/\mathbb{Z}w_1+\mathbb{Z}w_2$ has cplx charts induced by $\mathbb{C}$

$$\mathbb{C} \xrightarrow{\pi} \mathbb{C}/\mathbb{Z}w_1+\mathbb{Z}w_2$$



choose $W \ni 0$, s.t. $W+W \cap \mathbb{Z}w_1+\mathbb{Z}w_2 = \phi$

$(p+W) \underset{\longleftarrow}{\xrightarrow{\pi}} \pi(p+W)$ has inverse π^{-1}

Transition fctions are parallel transport.

Graph of holomorphic function

$w = f(z)$, f holomorphic

$\{ (z, w) \mid w = f(z) \}$ has

local chart given by

$(z, w) \mapsto z$

Similarly as real manifold

non degenerate equations gives rise

to submanifold

we study "Riemann surfaces"

given by equations

For example, $y^2 = (1-x^2)(1-k^2x^2)$

$k \neq 0, \pm 1$

For two variables $(z_1, z_2) \in U \subset \mathbb{C}^2$

$f(z_1, z_2)$ continuous function

holomorphic with respect to

z_1, z_2, (called holomorphic w.r.t two variables)

assume $\frac{\partial f}{\partial z_1} \neq 0$, $\frac{\partial f}{\partial z_2} \neq 0$

for $\{(z_1, z_2) \in U \mid f(z_1, z_2) = 0\}$

(or $\nabla f = (f_{z_1}, f_{z_2}) \neq (0, 0)$),

Then $\{f = 0\}$ has a structure

of Riemann surface

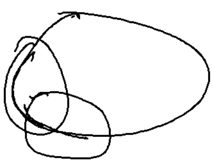
(Implicit function Theorem)

$$C = U \cap \{f = 0\}$$

$\frac{\partial f}{\partial z_1} \neq 0$ at $(a, b) \in C$

Then $\exists$ nbhd of $(a, b) \ni V$ and holomorphic function $g(z_2)$ defined on D_b open disc at b s.t. $V \cap \{f = 0\} =$

$$\{(g(z_2), z_2) \mid z_2 \in D_b\}$$



$V_1 \cap C$ $\qquad$ $V_2 \cap C$

(z_1, z_2) $\qquad$ (z_1, z_2)

$\swarrow$ $\qquad$ $\downarrow$

$z_2 \longmapsto z_1$

$z_2 \longmapsto g(z_2)$

$\boxed{\text{holomorphic}}$

Ex: $y^2 = (1-x^2)$

$$f(x,y) = x^2+y^2-1$$

$$\frac{\partial f}{\partial x} = 0 \Rightarrow x = 0$$

$$\frac{\partial f}{\partial y} = 0 \Rightarrow y = 0$$

$$(x,y) = (0,0) \Rightarrow f(0,0) = -1 \neq 0$$

Several cplx variables $\quad |z|<\varepsilon_1, \ |w|<\varepsilon_2$

$$f(z,w) = \frac{1}{(2\pi i)^2}\int\int \frac{f(\xi_1,\xi_2)\, d\xi_2\, d\xi_1}{(\xi_2-w)(\xi_1-z)}$$

$|\xi_1|=\varepsilon_1 \ |\xi_2|=\varepsilon_2$

$\left(= \text{power series expansion of } (z, w) \right)$

$\Rightarrow$ $f(z,w) = u + iv$, then

u, v are smooth functions of

$(z = a+bi,\ w = c+di)$ (a, b, c, d)

$$\frac{\partial f}{\partial z} = \frac{\partial u}{\partial a} + i\frac{\partial v}{\partial a} = \frac{\partial v}{\partial b} - i\frac{\partial u}{\partial b} = A + iB$$

so $$\frac{\partial(u,v)}{\partial(a,b)} = \begin{pmatrix} A & B \\ -B & A \end{pmatrix}$$

$\det = A^2 + B^2 \neq 0 \Leftrightarrow A + iB \neq 0$

Implicit function theorem $\Rightarrow$

locally $\begin{cases} u=0 \\ v=0 \end{cases}$ is given by

$$\begin{cases} c = c(a,b) \\ d = d(a,b) \end{cases}$$

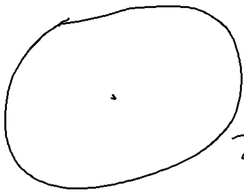
i.e. $z_2 = g(z_1)$

Only need to check g is holomorphic

(one way to apply chain rule to obtain $\frac{\partial g}{\partial \bar{z}_1} = 0$)

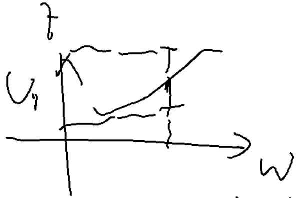
Another way Cauchy's formula

Recall $\frac{1}{2\pi i}\oint_{\partial U} \frac{f'(z)}{f(z)} =$ # of zeros of $f(z)$ in U



$$\frac{1}{2\pi i}\oint z \frac{f'(z)}{f(z)} = \text{sum of zeros}$$

$$\oint_{\partial U} z \frac{f_z(z,w)}{f(z,w)}\, dz \;=\; \underline{g(w)}$$



Fix $w=b$

$f(z,b) \neq 0$ at

∂U_1, $U_1 \ni a$ nbhd of a

and a is only zero of

$f(z, b)$ for $z \in U$

$$\frac{1}{2\pi i}\oint_{\partial U_1} \frac{f_z(z,w)}{f(z,w)}\, dz \text{ is } = N(w)$$

"well defined" for $w \in D_b$

$\left(|f(z,w)| > \varepsilon \text{ for } \begin{array}{l} w \in D_b \\ z \in \partial U_1 \end{array} \right)$

and continuous, integer-valued,

$= 1$ for $w = b$

so $N(w) = 1$ for $w \in D_b$

i.e. $f(z, w)$ has only one zero for every $w \in D_b$

and this zero point z is given

by $\frac{1}{2\pi i} \oint_{\partial U_1} z \frac{f_z(z, w)}{f(z, w)} dz = g(w)$

which is holomorphic w.r.t w