

IMSC 2048 HW10

Due 2026/4/9

April 2, 2026

1 Exercises

Normal extensions and separable extensions

Exercise 1 (Recognizing normal extensions). *Determine whether each of the following extensions is normal, and justify your answer.*

1. $\mathbb{Q}(\sqrt{2})/\mathbb{Q}$.
2. $\mathbb{Q}(\sqrt[3]{2})/\mathbb{Q}$.
3. $\mathbb{Q}(\sqrt[4]{2})/\mathbb{Q}(\sqrt{2})$.
4. $\mathbb{Q}(\sqrt[3]{2}, \omega)/\mathbb{Q}$, where $\omega = e^{2\pi i/3}$.
5. $\mathbb{Q}(\sqrt[3]{2}, \omega)/\mathbb{Q}(\omega)$, where $\omega = e^{2\pi i/3}$.

Exercise 2 (Intersection and compositum of normal extensions). *Let L_1/K and L_2/K be two normal extensions inside a common algebraic closure \bar{K} . (You can assume L_1 and L_2 are finite over K if you like, but it is not necessary.)*

1. *Show that $L_1 \cap L_2$ is normal over K .*
2. *Show that the subfield L_1L_2 generated by L_1 and L_2 is normal over K .*
3. *Is the analogue true for separable extensions? For Galois extensions?*

Exercise 3. *For each of the following extensions, compute $[L : K]$, $|\text{Hom}_K(L, \bar{K})|$, and $|\text{Aut}_K(L)|$, and determine which of the properties (normal, separable, Galois) hold.*

1. $\mathbb{Q}(\sqrt{2}, \sqrt{3})/\mathbb{Q}$.
2. $\mathbb{Q}(\sqrt[4]{2})/\mathbb{Q}$.
3. $\mathbb{F}_{p^n}/\mathbb{F}_p$. (Hint: The Frobenius endomorphism $x \mapsto x^p$.)
4. $\mathbb{Q}(\zeta_7)/\mathbb{Q}$, where $\zeta_7 = e^{2\pi i/7}$.

Galois correspondence: computation and structure

Exercise 4. Let L/K be a finite Galois extension. Prove that there are only finitely many intermediate fields M with $K \subseteq M \subseteq L$. If we only assume L/K is finite separable, does the same conclusion hold?

Exercise 5 (Galois correspondence for $\mathbb{Q}(\sqrt{2}, \sqrt{3})/\mathbb{Q}$). Let $L = \mathbb{Q}(\sqrt{2}, \sqrt{3})$.

1. Show that $\text{Gal}(L/\mathbb{Q}) \cong \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ by describing the four automorphisms explicitly.
2. List all subgroups of G and the corresponding intermediate fields.
3. Verify that every intermediate field is normal over \mathbb{Q} (and explain why this follows from the fact that G is abelian).

Exercise 6 (Compositum and intersection). Let L/K be a finite Galois extension and let M_1, M_2 be intermediate fields. Let M_1M_2 be subfield generated by M_1 and M_2 . Prove that:

1. $\text{Gal}(L/(M_1M_2)) = \text{Gal}(L/M_1) \cap \text{Gal}(L/M_2)$.
2. $\text{Gal}(L/(M_1 \cap M_2))$ is generated by $\text{Gal}(L/M_1) \cup \text{Gal}(L/M_2)$.

Apply this to $L = \mathbb{Q}(\sqrt{2}, \sqrt{3})$, $M_1 = \mathbb{Q}(\sqrt{2})$, $M_2 = \mathbb{Q}(\sqrt{3})$ and verify directly.

Exercise 7 (Fundamental theorem: degree formulas). Let L/K be a finite Galois extension with Galois group G , and let M be an intermediate field with corresponding subgroup $H = \text{Gal}(L/M)$. Show that: $[L : M] = |H|$ and $[M : K] = [G : H]$.

Optional

Exercise 8 (Galois correspondence for $x^4 - 2$). Compute the Galois group of $x^4 - 2$ over \mathbb{Q} . Find all intermediate fields. Which intermediate fields are normal over \mathbb{Q} ? (You may use Example ?? as a guide, or work independently.)

Exercise 9 (Galois correspondence for $x^3 - 2$). Let L be the splitting field of $x^3 - 2$ over \mathbb{Q} .

1. Show that $\text{Gal}(L/\mathbb{Q}) \cong S_3$ by describing the action of each automorphism on the roots $\sqrt[3]{2}$, $\sqrt[3]{2}\omega$, $\sqrt[3]{2}\omega^2$.
2. Draw the subgroup lattice of S_3 and the corresponding intermediate fields.
3. Which intermediate fields are Galois over \mathbb{Q} ? Verify using the normal subgroup criterion.