IMSC 2048 HW4
Due 2026/2/5

January 29, 2026

1 Excercies

1.1 Mandatory part

Excercise 1. *Is it true that the conjugacy classes of unitary group* $\mathrm{U}(2)$ *are determined by the trace and determinant? Prove your answer.*

Excercise 2.

Definition 1 (Orthogonal Group of Signature (p, q))**.** *Let* p, q *be non-negative integers. The orthogonal group of signature* (p, q)*, denoted by* $\mathrm{O}_{p,q}$*, is defined as the group of all linear transformations on* $\mathbb{R}^{p+q}$ *that preserve the bilinear form*

$$\langle x, y\rangle = x_1y_1 + x_2y_2 + \cdots + x_py_p - x_{p+1}y_{p+1} - \cdots - x_{p+q}y_{p+q},$$

for all $x, y \in \mathbb{R}^{p+q}$*.*

Let W *be the space of real trace-zero* 2×2 *matrices* $W = \{A \in M_{2\times 2}(\mathbb{R}) | trace(A) = 0\}$*.* W *has a basis* $\mathbf{B} = (w_1, w_2, w_3)$ *, where*

$$w_1 = \begin{bmatrix} 1 & \\ & -1 \end{bmatrix}, w_2 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, w_3 = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$$

1. *Show that the symmetric bilinear form defined by* $\langle A, A'\rangle = \mathrm{trace}\,(AA')$ *has signature* $(2, 1)$*. (Hint: use basis* $\mathbf{B}$*)*
2. *Prove that* $P \star A = PAP^{-1}$ *defines a linear group operation of* $\mathrm{SL}(2, \mathbb{R})$ *on the space* W*.*
3. *Use this operation to define a group homomorphism* $\varphi : \mathrm{SL}(2, \mathbb{R}) \to \mathrm{O}_{2,1}$*.*
4. *Prove the kernel of this homomorphism is* $\{\pm I\}$*.*

This construction actually shows that $\mathrm{SL}(2, \mathbb{R})$ *is a double cover of* $\mathrm{SO}^+_{2,1}$*, the connected component of* $\mathrm{O}_{2,1}$ *containing the identity matrix. Or equivalently,* $\mathrm{PSL}(2, \mathbb{R}) = \mathrm{SL}(2, \mathbb{R})/\{\pm I\}$ *is isomorphic to the spin group* $\mathrm{SO}^+(2, 1)$*. It is an* *interesting question to show that the orthogonal group* $\mathrm{O}_{2,1}$ *has four connected components and identify the geometry of each component.*

A similar excercise is to relate $\mathrm{SL}(2,\mathbb{C})$ *to the orthogonal group* $\mathrm{O}(3,1)$*. see Artin Algebra Chapter 9, 4.8*

Excercise 3. *Let* A *be the set of all* $n \times n$ *upper triangular matrices with real entries and all diagonal entries equal to* 1*. Find the Lie algebra of* A *and compute its dimension.*

Excercise 4. *Show that the intersection of symplectic group* $\mathrm{Sp}(2n,\mathbb{R})$ *and orthogonal group* $\mathrm{O}(2n,\mathbb{R})$ *in* $\mathrm{GL}(2n,\mathbb{R})$ *is isomorphic to the unitary group* $\mathrm{U}(n)$*. Here we use the embedding of these groups into* $\mathrm{GL}(2n,\mathbb{R})$ *as Section Examples of Lie groups and Lie algebras.*

Excercise 5. *Prove the Jacobi identity of Lie algebra* $\mathfrak{gl}(n,\mathbb{R}) = M_n(\mathbb{R})$ *using the properties of the matrix commutator. Here*

$$[A,B] = AB - BA.$$

1.2 Optional excercises

Excercise 6. *Prove that the Lie algebra of the symplectic group* $\mathrm{Sp}(2n,\mathbb{R})$ *is*

$$\mathfrak{sp}(2n,\mathbb{R}) = \{A \in M_{2n}(\mathbb{R}) : A^T J + JA = 0\},$$

where

$$J = \begin{pmatrix} \mathrm{O}_n & \mathrm{I}_n \\ -\mathrm{I}_n & \mathrm{O}_n \end{pmatrix}.$$

Excercise 7. *Prove the second and the third order term in the Baker-Campbell-Hausdorff formula when* X, Y *are elements of* $M_n(\mathbb{R})$*. That is, prove*

$$\exp(tX)\exp(tY) = \exp\left(t(X+Y) + \frac{1}{2}t^2[X,Y] + \frac{1}{12}t^3([X,[X,Y]] + [Y,[Y,X]]) + \cdots\right)$$

You may use the following expansion of logarithm:

$$\log(I + A) = A - \frac{1}{2}A^2 + \frac{1}{3}A^3 - \cdots$$