IMSC 2048 HW5
Due 2026/2/12

February 5, 2026

1 Excercies

1.1 Mandatory part

Excercise 1. *Let V be an irreducible representation of a finite group G over $\mathbb{C}$. Assume V is not the trivial representation. Prove that for any $v \in V$, we have $\sum_{g \in G} g \cdot v = 0$. (For cyclic groups, this is known to be an identity of roots of unity.)*

Excercise 2. *(Artin Algebra Chapter 10, 3.5) Let x be a generator of a cyclic group G of order p. Sending*

$$x \mapsto \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

defines a matrix representation $G \to \mathrm{GL}_2(\mathbb{F}_p)$. Prove that this representation is not the direct sum of irreducible representations.

Excercise 3. *(Artin Algebra Chapter 10, 3.4) Let $\langle \cdot, \cdot \rangle$ be a nondegenerate skew-symmetric form on a vector space V, and let ρ be a representation of a finite group G on V. Prove that the averaging process produces a G-invariant skew-symmetric form on V, and show by example that the form obtained in this way need not be nondegenerate.*

Excercise 4. *(Artin algebra Chapter 10, 2.2) Consider the standard two-dimensional complex representation of the dihedral group D_n. For which n is this an irreducible complex representation?*

Here the standard representation is given by the action of D_n as the group of symmetries of a regular n-gon in the plane, or equivalently, the representation defined by the matrices

$$r \mapsto \begin{pmatrix} \cos(2\pi/n) & -\sin(2\pi/n) \\ \sin(2\pi/n) & \cos(2\pi/n) \end{pmatrix}, \quad s \mapsto \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

where r is a rotation by $2\pi/n$ and s is a reflection.

Excercise 5. *(Artin Algebra Chapter 10, 3.1) Let G be a cyclic group of order 3. The matrix*

$$A = \begin{pmatrix} 0 & 1 \\ -1 & -1 \end{pmatrix}$$

has order 3, so it defines a matrix representation of G on $\mathbb{C}^2$. Use the averaging process to produce a G-invariant form from the standard Hermitian product $\langle X, Y \rangle = X^* Y$ on $\mathbb{C}^2$.

Excercise 6. *(Artin Algebra Chapter 10, 3.2) Let $\rho\colon G \to \mathrm{GL}(V)$ be a representation of a finite group G on a real finite-dimensional vector space V. Prove the following:*

1. *There exists a G-invariant, positive definite symmetric form $\langle \cdot, \cdot \rangle$ on V.*
2. *ρ is a direct sum of irreducible representations.*
3. *Every finite subgroup of $\mathrm{GL}_n(\mathbb{R})$ is conjugate to a subgroup of $\mathrm{O}(n)$.*

Excercise 7. *Show that the dual representation of an irreducible representation is also irreducible.*

1.2 Optional excercises

Excercise 8. *In this part, we prove the semisimplicity theorem (Maschke's theorem) for any representation of group G and field F such that the characteristic of F does not divide the order of G, using the averaging technique.*

Let V be a representation of G over F, and let W be a G-invariant subspace of V. Prove that there exists a complementary G-invariant subspace W' of V such that $V = W \oplus W'$. (Hint: start with any complementary subspace and then use averaging to construct a projection onto W whose kernel is W'.)

Excercise 9. *Show that the dual representation is isomorphic to the original representation for any finite group representation if and only if there exists a nondegenerate G-invariant bilinear form on the representation space.*