IMSC 2048 HW7
Due 2026/3/5

February 26, 2026

1 Exercises

1.1 Mandatory part

In this homework, we always work over the complex field $\mathbb{C}$ and finite dimensional representations of finite groups.

Excercise 1. *You can skip the first three questions if you already learned them in group theory last semester. Let $[G, G]$ be subgroup of G generated by all commutators $[g, h] = ghg^{-1}h^{-1}$ for $g, h \in G$.*

1. *Show that $[G, G]$ is a normal subgroup of G.*
2. *Show that $G/[G, G]$ is abelian.*
3. *Show that if H is abelian and $\phi\colon G \to H$ is a group homomorphism, then $[G, G] \subset \ker \phi$.*
4. *Prove that there is an one-to-one correspondence between the one-dimensional representations of G and the irreducible representations of $G/[G, G]$.*
5. *Show that the commutator subgroup $[G, G]$ is the intersection of kernels of all one-dimensional characters of G.*

Excercise 2. *From class, we know that the character χ_{reg} of regular representation satisfies $\chi_{reg}(g) = 0$ if $g \neq e$. There is an inverse of this proposition. Let χ be a character of G and satisfies $\chi(g) = 0$ if $g \neq e$. Prove that the corresponding representation is the direct sum of several copies of regular representation, i.e. $\rho \cong \rho_{reg} \oplus \rho_{reg} \cdots \oplus \rho_{reg}$ for some integer n.*

Excercise 3. *In this question, you will find the character table of A_4 the group of even permutations of S_4. Alternating group A_4 is a subgroup of S_4 and has 4 conjugacy classes*

$$\{(1)\}, \{(12)(34), (23)(14), (13)(24)\}, \{(123), (142), (134), (243)\}, \{(132), (124), (143), (234)\}.$$

1. *Prove that* $K = \{(1), (12)(34), (23)(14), (13)(24)\}$ *is a normal subgroup of* S_4.

2. *Prove that* A_4/K *is cyclic group of order* 3. *(You can use the fact that any group of prime order is a cyclic group, think about why this is true.)*

3. *Use lifting to find all the irreducible characters of* A_4.

4. *(Optional) Compare this with character table of* S_4 *and describe all the irreducible representations of* A_4.

Excercise 4. *Let* G *be a finite group and* $g \in G$. *Prove that* g *and* g^{-1} *are in the same conjugacy class if and only if* $\chi(g)$ *is in* $\mathbb{R}$ *for all characters* χ.

Excercise 5. *Below is a partial character table for* G. *One conjugacy class and one row are missing. Here* e *is the identity element. The numbers in the top row mean the numbers of elements in each conjugacy class.*

$$\begin{array}{c|ccccc} & (1) & (1) & (2) & (2) & (3) \\ & e & u & v & w & x \\ \hline \chi_1 & 1 & 1 & 1 & 1 & 1 \\ \chi_2 & 1 & 1 & 1 & 1 & -1 \\ \chi_3 & 1 & -1 & 1 & -1 & \sqrt{-1} \\ \chi_4 & 1 & -1 & 1 & -1 & -\sqrt{-1} \\ \chi_5 & 2 & 2 & -1 & -1 & 0 \end{array} \tag{1}$$

1. *Complete the character table. (Hint: the number of elements in each conjugacy class divides the order of* G.*)*

2. *Determine the normal subgroups of* G.

3. *Find the orders of representative elements in each conjugacy class.*

Excercise 6. *Let* $\mathbb{C}[G]$ *be the underlying vector space for regular representation, i.e. the elements in* $\mathbb{C}[G]$ *are formal linear combinations of elements in* G. *Let* V *be a group representation of* G. *Prove that the map* $F\colon Hom_G(\mathbb{C}[G], V) \to V$ *defined by* $F(f) = f(e)$ *gives a linear space isomorphism. Use this isomorphism to give a new proof of the fact that the multiplicity of an irreducible representation* W *in regular representation is equal to* $\dim W$.

1.2 Optional problems

Excercise 7. *Write down the character table for group* A_5, *the group of even permutations on* 5 *elements. (Hint:* A_5 *has* 5 *conjugacy classes and thus has* 5 *irreducible representations.)*