

# IMSC 2048 HW8

## Due 2026/3/26

March 19, 2026

### 1 Exercises

#### 1.1 Mandatory part

**Exercise 1.** Show that cyclotomic polynomials  $\Phi_p(x) = \frac{x^p-1}{x-1}$  are irreducible over  $\mathbb{Q}$  for all prime numbers  $p$ .

**Exercise 2.** Prove that the polynomial  $x^4 + 3x + 3$  is irreducible polynomial over the field  $\mathbb{Q}[\sqrt[3]{2}]$ .

**Exercise 3.** Find the degree of field extension

1.  $[\mathbb{Q}[\sqrt{p}, \sqrt{q}] : \mathbb{Q}]$  where  $p$  and  $q$  are two distinct prime numbers.
2.  $[\mathbb{Q}[\sqrt[3]{2}, \sqrt{2}] : \mathbb{Q}]$

**Exercise 4.** Find the irreducible polynomial of  $\sqrt[3]{2} + \sqrt{3}$  over  $\mathbb{Q}$ .

**Exercise 5.** Find whether  $(7)$  is a prime ideal in  $\mathbb{Z}[\sqrt{-3}]$ .

**Exercise 6.** Find the splitting field  $L$  of  $x^8 - 2$  over  $\mathbb{Q}$  (Write  $L$  in the form of  $\mathbb{Q}(\alpha, \beta, \dots)$ ), and compute  $[L : \mathbb{Q}]$ .

**Exercise 7.** Let  $L$  be the splitting field of  $x^3 - 1$  over  $\mathbb{F}_2$ . Compute  $[L : \mathbb{F}_2]$ .

**Exercise 8.** We call two extensions  $K_1$  and  $K_2$  of  $F$  isomorphic if there exists a field isomorphism  $\varphi: K_1 \rightarrow K_2$  such that  $\varphi|_F: F \rightarrow F$  is identity. Assume  $\text{char}(F) \neq 2$ . Classify the isomorphism classes of degree-two extensions (quadratic extensions) of  $F$ . (Find an identification with  $F^\times / (F^\times)^2 - \{\bar{1}\}$ , and explain the structure of this group  $F^\times / (F^\times)^2$  using prime factorization when  $F = \mathbb{Q}$ .)

## 1.2 Optional questions

**Exercise 9.** Let  $p_1, \dots, p_n$  be distinct primes. Prove that  $[\mathbb{Q}(\sqrt{p_1}, \dots, \sqrt{p_n}) : \mathbb{Q}] = 2^n$ .

**Exercise 10.** Consider quadratic extensions  $K$  of  $F$  with  $\text{char } F = 2$ . Prove that either  $K = F[\alpha]$  with  $\alpha^2 \in F$  and  $\alpha \notin F$  or  $K = F[\alpha]$  with  $\alpha^2 - \alpha \in F$  and  $\alpha \notin F$ . Can two cases of two different types above be isomorphic? Classify the isomorphism classes for quadratic extensions of  $F$  using certain quotient groups of  $F^\times / (F^\times)^2$  and  $(F, +) / \{a^2 - a \mid a \in F\}$  similar to question 8.

**Exercise 11.** Let  $p$  be a prime integer. Show that  $x^p - x - a$  is irreducible over  $\mathbb{F}_p$  for any  $a \in \mathbb{F}_p^\times$ . (Hint: If it has a root, then it splits by the shifting one root.)