IMSC 2048 HW9
Due 2026/4/2

March 26, 2026

1 Exercise

Excercise 1. *Let F be a field with finite characteristic p, and $f(x) \in F[x]$ be an irreducible polynomial. Prove that there exists a unique integer $e \geq 0$ and a unique irreducible separable polynomial $g(x) \in F[x]$ such that $f(x) = g(x^{p^e})$.*

Excercise 2. *Let F be a field. Let $f(x)$ and $g(x)$ be two coprime polynomials in $F[x]$.*

1. *Show that $f(x)y - g(x)$ is irreducible in $F[x, y]$, and also irreducible in both $F(y)[x]$ and $F(x)[y]$.(Hint: use Gauss's lemma)*
2. *Prove that $F(x)$ is a finite extension of $F(f(x)/g(x))$ and its degree is* $\max\{\deg f, \deg g\}$.

Excercise 3. *Let K/F be a field field extension. Denote by $Aut_F(K)$ the automorphisms σ of K such that $\sigma|_F = Id_F$. Prove that*

$$Aut_F(F(x)) = \{x \mapsto \frac{ax+b}{cx+d} \mid \det \begin{bmatrix} a & b \\ c & d \end{bmatrix} \neq 0\}$$

and $Aut_F(F(x)) \cong PGL(2, F) = GL(2, F)/\{\lambda I_2 \mid \lambda \in F^\times\}$ as groups.

Excercise 4. *Construct a splitting field for $x^5 - 2$ over $\mathbb{Q}$ by writing down a set of generators. What is its degree over $\mathbb{Q}$?*

Excercise 5. *Let $F \subset K \subset L$ be finite field extensions. Then L/F is separable if and only if K/F is separable and L/K is separable.*

Excercise 6 (Application of uniqueness of splitting fields)**.** *Let p be a prime and consider $f(x) = x^{p^2} - x \in \mathbb{F}_p[x]$.*

A useful fact about finite subgroups in $F^\times$ is that they are cyclic.

1. *Show that the splitting field of f over $\mathbb{F}_p$ is a field with p^2 elements, denoted by $\mathbb{F}_{p^2}$.*

2. *Using the uniqueness of splitting fields, show that* $\mathbb{F}_{p^2}$ *is (up to isomorphism) the unique field with* p^2 *elements.*

3. *More generally, use the same argument to show that for any* $n \geq 1$, $\mathbb{F}_{p^n}$ *is the unique field (up to isomorphism) with* p^n *elements.*

Excercise 7. *Let* L/K *be an algebraic field extension. Show that all the separable elements in* L *over* K *form a subfield* L_s *of* L *containing* K*. (Optional: if* $L \neq L_s$*, then* L *is not separable over* L_s*.)*

Excercise 8 (Degree bounds for splitting fields)**.** *Let* $f(x) \in K[x]$ *be a polynomial of degree* n *with splitting field* L*. Show that* $[L : K]$ *divides* $n!$.

1.1 Optional

Excercise 9 (Perfect fields)**.** *Recall that a field* K *is* ***perfect*** *if every algebraic extension of* K *is separable.*

1. *Show that a field of characteristic* 0 *is perfect.*

2. *Show that a field* K *of characteristic* $p > 0$ *is perfect if and only if the Frobenius map* $\varphi\colon K \to K$, $a \mapsto a^p$*, is surjective (i.e. every element of* K *has a* p*-th root in* K*).*

3. *Deduce that every finite field is perfect.*

4. *Give an example of an imperfect field.*

Excercise 10. *Let* $\overline{F}$ *be the algebraic closure of* F *and* F^{sep} *be the subfield of* $\overline{F}$ *consisting of all elements* α *such that* $F(\alpha)/F$ *is separable. Prove that* F^{sep} *is the maximal separable extension of* F *in* $\overline{F}$ *in the sense that any separable extension of* F *in* $\overline{F}$ *can be embedded into in* F^{sep}*, and* F^{sep} *has no finite separable extension of degree greater than* 1*. (Optional: show that the these properties uniquely determine* F^{sep}/F *up to isomorphism. After next class on normal extensions, you can ask is* F^{sep}/F *normal?)*