IMSC 2048 HW2
Due 2026/1/22

January 15, 2026

1 Excercises

1.1 Useful Exercises

You are required to submit the solutions to problems in this subsection.

Problem 1. *Prove that any skew-symmetric matrix $A \in M_n(\mathbb{R})$ can be orthogonally similar to a block diagonal matrix with blocks of the form*

$$\begin{pmatrix} 0 & -\lambda \\ \lambda & 0 \end{pmatrix}$$

and possibly a 0 *block if n is odd. Use this to show that any skew-symmetric matrix over $\mathbb{R}$ is congruent to a block diagonal matrix with blocks of the form*

$$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

and possibly some 0 *block.*

Problem 2. *Let V be the linear space consisting of all skew-symmetric real matrices of order n.*

1. *For any $A \in V$, prove that $I + A$ is invertible.*
2. *For any $A \in V$, define $f(A) = (I - A)(I + A)^{-1}$. Prove that $f(A)$ is an orthogonal matrix.*
3. *Give a characterization of the image of $f : V \to \mathrm{O}(n)$ in terms of eigenvalues, that is, which matrices can be written in the form $(I - A)(I + A)^{-1}$ for some $A \in V$.*

Problem 3. *Let A be 2×2 real symmetric matrix*

$$A = \begin{pmatrix} a & b \\ b & c \end{pmatrix}.$$

Write down an orthogonal matrix Q which diagonalizes A in terms of a, b, c.

Problem 4. *Consider the groups* $\mathrm{O}(2)$*, its subgroup* $\mathrm{SO}(2)$ *and group* $\mathrm{SO}(3)$*. Determine whether the following statements are correct. If correct, prove it; if incorrect, provide a counterexample:*

1. *Two elements in the group* $\mathrm{O}(2)$ *are conjugate if and only if they have the same trace.*

2. *Two elements in the group* $\mathrm{SO}(2)$ *are conjugate in the group* $\mathrm{SO}(2)$ *if and only if they have the same trace.*

3. *Two elements in the group* $\mathrm{SO}(2)$ *are conjugate in the group* $\mathrm{O}(2)$ *if and only if they have the same trace.*

4. *Two elements in the group* $\mathrm{SO}(3)$ *are conjugate if and only if they have the same trace.*

Problem 5 (Cartan–Dieudonné theorem)**.** *Prove that any orthogonal transformation of Euclidean space* $(V, \langle\cdot,\cdot\rangle)$ *can be expressed as a composition of at most* $\dim V$ *reflections.*

(The nontrivial part of the original theorem is to show this also holds for any non-degenerate symmetric bilinear form over a field of characteristic not equal to 2*.)*

Problem 6 (Courant–Fischer–Weyl Min-Max Principle)**.** *You may choose to prove either part (1) or part (2).*

1. *Let* $(E, \langle\cdot,\cdot\rangle)$ *be an* n*-dimensional real inner product space. Suppose* T *is a self-adjoint transformation on* E *with real eigenvalues* $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$*. Prove that the eigenvalues of* T *can be characterized by the following min-max method:*

   $$\lambda_k = \min\left\{\max\left\{\langle T(x), x\rangle : x \perp W_k, |x| = 1\right\} : W_k \subset E \textit{ is a subspace}, \dim W_k = k-1\right\}$$

   Here, for a fixed $(k-1)$*-dimensional subspace* W_k*, we first compute the maximum value*

   $$\max\left\{\langle T(x), x\rangle : x \perp W_k, |x| = 1\right\}.$$

   Then we vary W_k *over all* $(k-1)$*-dimensional subspaces and take the minimum of these maximum values.*

2. *Alternatively, you may prove the following special case: Let* A *be an* $n \times n$ *real symmetric matrix and* v *be an arbitrary* n*-dimensional real column vector, where* $|v|$ *denotes the vector length under the standard inner product. Let* $\lambda_1, \lambda_2, \cdots, \lambda_n$ *be all eigenvalues of* A*. Prove that:*

   $$|Av| \leq \max\{|\lambda_1|, |\lambda_2|, \cdots, |\lambda_n|\}|v|.$$

1.2 Optional problems

You do not need to hand in these problems, but you are encouraged to discuss and try them.

Problem 7 (Outer automorphisms of $\mathrm{SO}(n,\mathbb{R})$)**.** *An automorphism of a group G is called an* inner automorphism *if it is of the form $g \mapsto hgh^{-1}$ for some fixed $h \in G$. An automorphism which is not inner is called an* outer automorphism. *Consider the automorphism of* $\mathrm{SO}(n,\mathbb{R})$ *defined by $A \mapsto PAP^{-1}$ where $P \in \mathrm{O}(n,\mathbb{R})$ with $\det P = -1$. Is this an inner automorphism or an outer automorphism? Prove your answer. (The answer may depend on n.)*

Problem 8 (Challenge Problem)**.** *You will obtain a standard form for Lorentz transformations on $\mathbb{R}^4$. Let e_i $(i = 1, \dots 4)$ be the standard basis for $\mathbb{R}^4$. Consider the symmetric bilinear on $\mathbb{R}^4$ defined by*

$$\langle x, y\rangle = x_1y_1 + x_2y_2 + x_3y_3 - x_4y_4.$$

A basis f_i $(i = 1, \dots 4)$ of $\mathbb{R}^4$ is called orthonormal if

$$\langle f_1, f_1\rangle = \langle f_2, f_2\rangle = \langle f_3, f_3\rangle = 1, \quad \langle f_4, f_4\rangle = -1, \quad \langle f_i, f_j\rangle = 0 \text{ if } i \neq j.$$

Suppose T is a Lorentz transformation on $\mathbb{R}^4$, that is, T is a linear transformation such that

$$\langle Tx, Ty\rangle = \langle x, y\rangle$$

for all $x, y \in \mathbb{R}^4$. Prove that there exists an orthonormal basis of $\mathbb{R}^4$ such that the matrix of T is block diagonal with blocks of the following types:

1. *A block of order 1 with entry ± 1.*
2. *A block of order 2 of the form*

$$\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}.$$

3. *a block of order 2 of the form*

$$\pm\begin{pmatrix} \cosh\theta & \sinh\theta \\ \sinh\theta & \cosh\theta \end{pmatrix} \quad \text{or} \ \pm\begin{pmatrix} \cosh\theta & \sinh\theta \\ -\sinh\theta & -\cosh\theta \end{pmatrix}.$$

4. *A block A of order 3 with eigenvalue $\lambda = \pm 1$ so that $(A - \lambda I)^3 = 0$ but $(A - \lambda I)^2 \neq 0$.*

Problem 9. *If the Lorentz transformation T in Problem 8 is replaced by a transformation satisfying*

$$\langle Tx, y\rangle = -\langle x, Ty\rangle$$

can you obtain a similar result? State the result and prove it.

Problem 10 (Cauchy Interlacing Theorem)**.** *Let A be an $n \times n$ real symmetric matrix, and let B be an $m \times m$ principal submatrix of A, where $m < n$. If the eigenvalues of A are $\lambda_1 \geqslant \lambda_2 \geqslant \cdots \geqslant \lambda_n$, and the eigenvalues of B are $\mu_1 \geqslant \mu_2 \geqslant \cdots \geqslant \mu_m$, then for all $1 \leqslant i \leqslant m$, we have*

$$\lambda_i \geqslant \mu_i \geqslant \lambda_{i+n-m}.$$

(Hint: Use the Courant-Fischer-Weyl min-max principle from Problem 6.)

Problem 11 (Sylvester's Criterion)**.** *Use the Cauchy interlacing theorem to prove Sylvester's criterion: A symmetric matrix is positive definite if and only if all its leading principal minors are positive.*