

香 港 中 文 大 學
The Chinese University of Hong Kong

二零二五至二六年度下學期科目考試
Course Examination 2nd Term, 2025-26

科目編號及名稱
Course Code & Title : IMSC 2048

時間
Time allowed : 3 小時/hours 0 分鐘/minutes

學號
Student I.D. No. : _____

座號
Seat No. : _____

Question 1 (10 marks)

Compute the QR -decomposition of the following matrix

$$A = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix}.$$

In other words, find an orthogonal matrix Q and an upper triangular matrix R such that $A = QR$ and the diagonal entries of R are positive.

Question 2 (10 marks)

Let G be a finite group and operates on a finite set X . Let V be the vector space over \mathbb{C} with basis X , i.e., $V = \mathbb{C}^X$. Then the G -action on X induces a linear representation of G on V . Show that number of orbits of G on X is equal to multiplicity of the trivial representation in the irreducible decomposition of V .

Question 3 (10 marks)

Consider the following two field extensions:

1. $F = \mathbb{F}_2(t)$, $K = F[x]/(x^2 + x + t)$;

2. $F = \mathbb{F}_2(t)$, $L = F[x]/(x^2 - t)$.

Determine whether the extensions K/F and L/F are separable or inseparable, normal or non-normal, and explain your reasoning.

Question 4 (15 marks)

Let $G = S_3$ the permutation group of three objects. Answer the following questions:

1. List the conjugacy classes of S_3 .
2. Write down the full character table of S_3 .
3. Consider the restrictions of the irreducible representations of S_3 to the cyclic subgroup C_3 generated by the 3-cycle (123) . Write down the character table of C_3 and decompose the restricted representations into irreducible representations of C_3 .

Question 5 (10 marks)

Consider the bilinear form defined on matrix space $M_n(\mathbb{R})$ by $\langle A, B \rangle = \text{tr}(AB)$. Determine the signature of this bilinear form (positive index of inertia and negative index of inertia). (You can get partial credit for the special cases $n = 1, 2, 3$.)

Question 6 (15 marks)

Let K be the splitting field of the polynomial $x^8 - 1$ over \mathbb{Q} . Let $G = \text{Gal}(K/\mathbb{Q})$ be the Galois group of K over \mathbb{Q} .

1. Determine the order of G .
2. Find all intermediate fields between \mathbb{Q} and K , and write them in the form of $\mathbb{Q}(\alpha)$ for some $\alpha \in K$.
3. Consider the operation of G on K as a vector space over \mathbb{Q} . Then this defines a \mathbb{Q} -representation of G . What is the character of this representation? Decompose it into irreducible \mathbb{C} -characters of G .

Question 7 (15 marks)

Find the degree of extensions $[K : F]$, determine whether the extension K/F is a Galois extension, and explain your reasoning.

1. $F = \mathbb{Q}$, $K = \mathbb{Q}(\sqrt[3]{2})$;
2. $F = \mathbb{C}(x^3 + \frac{1}{x^3})$, $K = \mathbb{C}(x)$;
3. $F = \mathbb{Q}$, $K = \mathbb{Q}(\sqrt{2} + \sqrt{3})$.

Question 8 (15 marks)

Consider polynomials $f(x) = x^5 - 4x - 2 \in \mathbb{Q}[x]$. We want to determine the Galois group of f over \mathbb{Q} . There are several facts that you can use without proof:

- $f = 0$ has three real roots and two non-real roots in \mathbb{C} .
- $f(x)$ modulo 3 is irreducible in $\mathbb{F}_3[x]$ and hence f is irreducible over \mathbb{Q} .
- $f(x)$ modulo 7 factors as $(x^2 + 3x + 6)(x^3 + 4x^2 + 3x + 2)$ in $\mathbb{F}_7[x]$ with two distinct irreducible factors of degree 2 and 3 respectively.

Let K be the splitting field of f over \mathbb{Q} and $G = \text{Gal}(K/\mathbb{Q})$ be the Galois group of f over \mathbb{Q} .

1. Find the degree of the extension $[K : \mathbb{Q}]$.
2. Determine the isomorphism class of G as a subgroup of S_5 and justify your answer.
3. Is the equation $f(x) = 0$ solvable by radicals over \mathbb{Q} ? Justify your answer.

Write your answers here