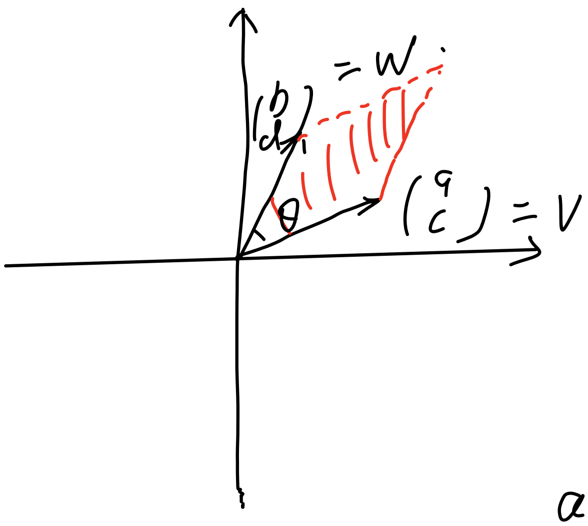
# 行列式：

2×2 矩阵是否可逆 $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$

$c, d \neq 0$， $\frac{a}{c} \neq \frac{b}{d}$， $\boxed{ad - bc \neq 0}$ 对 $c, d = 0$ 也有效.

另一个几何含义： $|ad - bc| =$ 平行四边形面积




为什么： $\cos\theta = \dfrac{ab + cd}{\sqrt{a^2 + c^2}\sqrt{b^2 + d^2}}$

$$面积 = \sqrt{1 - \cos^2\theta} \cdot \sqrt{a^2 + c^2}\sqrt{b^2 + d^2}$$

$$= \sqrt{(ad - bc)^2} = |ad - bc|$$

$ad - bc > 0$， $w$ 在 $v$ 左边， $0 < \theta < \pi$

$ad - bc < 0$， $w$ 在 $v$ 右边， $0 < \theta < \pi$.

$ad-bc$ 是"有向"面积

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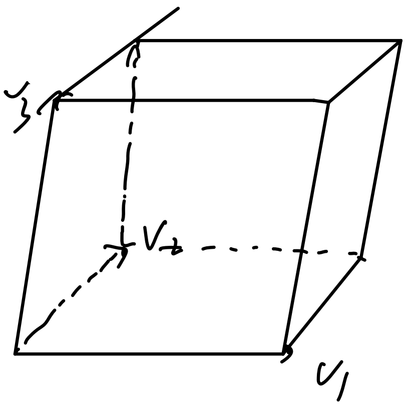
高维推广. parallelepiped 体积




= 底面积 · 高.

(有向) 体积

$(v_1, v_2, v_3) \longmapsto$ 体积

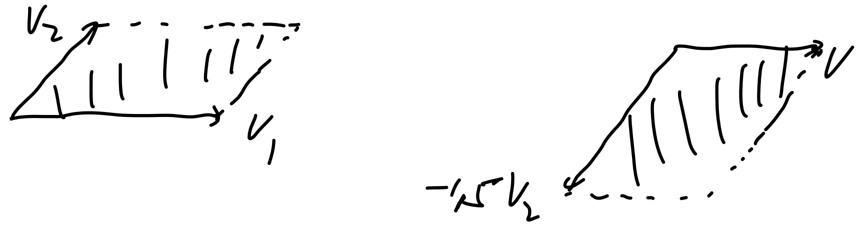
一种定义方式: $V = \mathbb{R}^n$ (利用线性结构)

定义一个函数 $f: \underbrace{V \times V \times \cdots \times V}_{n个} \to F$

$(v_1, v_2, \cdots v_n) \longmapsto f(v_1, \cdots v_n)$

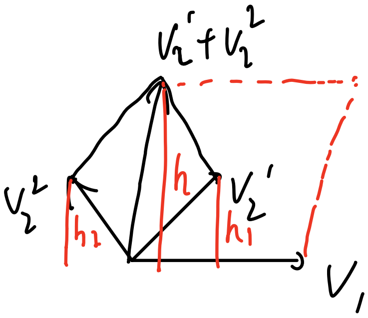
满足条件: ① $f(v_1, v_2, \cdots, cv_i, \cdots v_n)$

$= c f(v_1 \cdots v_n)$



② $f(v_1, v_2, \cdots v_i^1 + v_i^2, \cdots v_n)$

$= \underline{f(v_1, v_2 \cdots v_i^1, \cdots v_n)} + \underline{f(v_1, v_2, \cdots v_i^2, \cdots v_n)}$

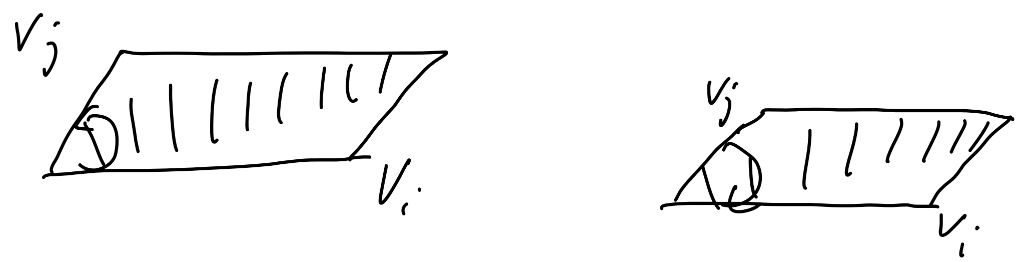



$\boxed{h = h_1 + h_2}$

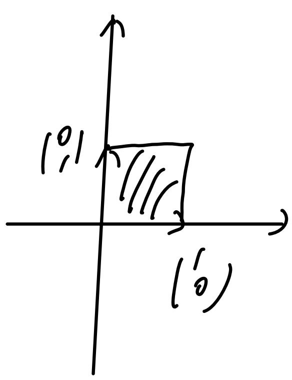
(图形拼凑)

③ $f(v_1, v_2, \cdots v_i, \cdots v_j, \cdots v_n)$

$= \underline{-f(v_1, v_2 \cdots v_j, \cdots v_i, \cdots v_n)}$




④ (normalization) $f(e_1 \cdots e_n) = 1.$



①. ② (多)线性. ③ 反对称.

定理(定义) $f$ 存在且唯一. $[V_1 \cdots V_n] = A$.

$$f(V_1 \cdots V_n) \overset{\text{定义}}{=} |A| = \det A.$$

一些基本性质:

③'. $f(V_1 \cdots V_i; \cdots V_j; \cdots V_n) = 0$

如果 $V_i = V_j$.

③ ⇒ ③'. $f(V_1 \cdots V_n) = -f(V_1 \cdots V_n)$

$(V_i = V_j)$

$\Rightarrow f(V_1 \cdots V_n) = 0$ $(2 \neq 0)$.

③' ⇒ ③

+②

$$f(V_1 \cdots V_i + V_j, \cdots V_i + V_j; \cdots V_n) = 0$$

$$= f(V_1 \cdots V_i, \cdots V_i, \cdots V_n) = 0$$

$$+ f(V_1 \cdots V_j, \cdots V_j, \cdots V_n) = 0$$

$$+ f(V_1 \cdots V_i, \cdots V_j, \cdots V_n)$$

$$+ f(V_1 \cdots V_j; \cdots V_i, \cdots V_n)$$

一些基本性质：

① 如果 $A$ 某一列为 $0$，则 $|A| = 0$

② $\det(AB) = (\det A)(\det B)$

③ $\det A = \det A^T$

④ $A$ 上三角阵 $\begin{bmatrix} a_1 & & * \\ & \ddots & \\ 0 & & a_n \end{bmatrix}$. $|A| = a_1 a_2 \cdots a_n$

⑤ $\det A \neq 0 \Leftrightarrow \underbrace{A \text{ 可逆} \Leftrightarrow \operatorname{rk} A = n}$

⑥ $A$ 准上三角阵 $\left[\begin{array}{c|c} A_1 & * \\ \hline 0 & A_2 \end{array}\right]$

$$\det A = \det A_1 \cdot \det A_2.$$

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证明：①. $f(v_1, \cdots, c \cdot 0, \cdots v_n) = \underline{c \cdot f(v_1 \cdots 0 \cdots v_n)}$

取 $c = 0$. $\Rightarrow f(v_1 \cdots 0, \cdots v_n) = 0$.

② 对初等矩阵 $E$，计算 $\det E \neq 0$

$\boxed{c \neq 0}$
$$\left|\begin{bmatrix} 1 & & & & \\ & \ddots & & & \\ & & c & & \\ & & & \ddots & \\ & & & & 1 \end{bmatrix}\right| = c \cdot f(e_1 \cdots e_n) = c$$

$$\begin{array}{c} \left|\begin{matrix} 1 & & & & \\ & \ddots & & & \\ & & 1 & c & \\ & & & 1 & \\ & & & & \ddots \quad 1 \end{matrix}\right| \\ \quad\quad\; i\text{列} \;\; j\text{列} \end{array} = f(e_1, e_2 \cdots e_j + c e_i, \cdots e_n)$$
$$= f(e_1, e_2 \cdots e_j \cdots e_n) + c\, f(e_1, e_2 \cdots \underset{e_j}{e_i} \cdots e_n)$$
$$= 1$$

$$\left|\begin{matrix} 1 & & & & \\ & \ddots & & & \\ & & 0 & 1 & \\ & & 1 & 0 & \\ & & & & \ddots \quad 1 \end{matrix}\right| = (-1)$$

对 $|A \cdot \underset{E\ \text{初等矩阵}}{E}| = |A| \cdot |E|$ ← 由定义中的性质

一般有 $A = A_1 E_1 E_2 \cdots E_k$

($A_1$ rref)

$\begin{cases} \text{rk } A < n, & A_1 \text{ 有 } 0 \text{ 列}. \quad |A| = 0 \\ \text{rk } A = n, & \text{则 } A_1 = I_n \ldots \end{cases}$

$|A| = |E_1| \cdots |E_k|$.

$\det(AB) = \begin{cases} B \text{ 可逆}, & A \cdot \underset{\text{初等}}{E_1} \cdots E_k \\ B \text{ 不可逆}, & \text{rk}(AB) < n \end{cases}$

$(\text{rk}(AB) \leq \text{rk}(B))$

$|AB| = 0, \quad |B| = 0$

③ $\det A = \det A^T$

$A = A_1 E_1 \cdots E_k$

$|A| = \underline{|A_1|} \cdot |E_1| \cdots |E_k|$.

$rk(A_1) < n$, $rk(A) < n \Rightarrow |A| = 0 = |A^T|$
$rk(A^T) < n$

$A_1 = I_n$. $|A| = |E_1| \cdots |E_k|$

易验证. $\underline{|E_i^T| = |E_i|}$

$A^T = E_k^T \cdots E_1^T$

④ $a_i$ 有 0, 则 $\underline{rk(A) < n}$ $|A| = 0$

$a_i$ 无 0, 用 $A = \underbrace{E_1 \cdots E_k}_{\text{第一种}} \underbrace{E_{k+1} \cdots E_l}_{\text{第二种}}$

$\Rightarrow |A| = a_1 \cdots a_n$.

⑤ ✓.

⑥ $\left[\begin{array}{c|c} A_1 & * \\ \hline 0 & A_2 \end{array}\right]$ 和④同样.

计算技巧：① 行变换（上三角阵）（普遍性）

② 行展开（3.）

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ a_{31} & & & \\ \vdots & & & \\ a_{n1} & \cdots & & a_{nn} \end{bmatrix}$$

$v_1 \quad v_2 \cdots v_n$

$f(v_1, v_2 \cdots v_n)$

$= f(a_{11} \cdot e_1 + a_{21} \cdot e_2, \cdots + a_{n1} e_n, v_2, \cdots v_n)$

$= a_{11} f(e_1, v_2, \cdots v_n)$

$+ a_{21} f(e_2, v_2, \cdots v_n)$

$+ a_{i1} \boxed{f(e_i, \boxed{v_2, \cdots v_n})} + \cdots$

$\boxed{f(e_i, v_2', \cdots v_n')}$

$\boxed{v_2 = a_{i2} \cdot e_i + v_2'}$

$v_2'$ 写作 $e_1 \cdots e_n$ 线性组合时只有出现 $e_1 \cdots \hat{e_i} \cdots e_n$

$$\underline{f(e_i}, \underline{v_2' \cdots v_n'})$$

$$v_2' \cdots v_n' \in \underline{\mathrm{span}(e_1 e_2 \cdots \hat{e_i}, \cdots e_n)} = W$$

定义一个新的函数。

$$g: \underbrace{W \times W \cdots \times W}_{(n-1)\text{个}} \longrightarrow \mathbb{R}$$

$$(w_1 \cdots w_{n-1}) \longmapsto \underline{f(e_i}, \underline{w_1 \cdots w_{n-1}})$$

满足所有（除了）

$$g(e_1 \cdot \hat{e_i}\, e_n) = f(e_i, e_1 \cdots e_n)$$

$$= \underline{(-1)^{i+1}} \cdot f(e_1 \cdots e_n)$$

$$= (-1)^{i+1} \cdot |$$

$$\Rightarrow f(e_i, \underline{v_2' \cdots v_n'}) = (-1)^{i+1} \cdot \begin{vmatrix} a_{12} \\ a_{22} \\ \cancel{a_{i2}} \\ a_{n2} \end{vmatrix}$$

去掉i行

(3)

$M_{ij}$ = A 去掉 i行, j列.

$$|A| = (-1)^{1+1} a_{11} \cdot |M_{11}| + (-1)^{1+2} a_{21} \cdot |M_{21}|$$

$$= \sum_{i} \left(a_{i1} \cdot |M_{i1}|\right) (-1)^{i+1}$$

一般: $|A| = \sum_{j} (-1)^{i+j} a_{ij} \, |M_{ij}|$

$$\begin{vmatrix} & & & & \\ a_{i1} & a_{i2} & \cdots & a_{ij} & a_{in} \\ & & & & \end{vmatrix}$$

记 $A_{ij} = (-1)^{i+j} |M_{ij}|$

则 $|A| = \sum_{j=1}^{n} a_{ij} A_{ij}$

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存在性. 唯一性 ⇐ 行展开 + 归纳法

（抽象一点的形式）

$V$ 是 $n$ 维线性空间，满足 ①. ②. ③ 的

$\{f: V\times\cdots\times V\to\mathbb{R}\}$ 作成线性空间 $\Lambda^n V^*$

只需证明 $\boxed{\dim \Lambda^n V^* = 1}$ $f\in\Lambda^n V^*$, $f\neq 0$

$f(v_1\cdots v_n)\neq 0$ 对任一组基 $v_1\cdots v_n$.

对 $n$ 作归纳：

取 $u\neq 0$, $u\in V$ $\quad$ $\mathrm{span}\, u$ 有补空间 $W$

$V=\mathrm{span}\, u\oplus W$

构造：$T:\Lambda^n V^*\to\Lambda^{n-1}W^*$

$f\longmapsto f'$ 定义为

$$f'(w_1\cdots w_{n-1}) = f(u, w_1\cdots w_{n-1})$$

想要证 $T$ 是同构.

① $\ker T=\{0\}$. $T(f)=0$ 函数.

$f(v_1 \cdots v_n)$ $v_i \in V$

$\boxed{v_i = a_i u + w_i}$

$f(v_1 \cdots v_n) = f(a_1 u + w_1, a_2 u + w_2, \cdots, a_n u + w_n)$

$= a_1 f(u, w_2 \cdots w_n)$

$+ a_2 f(w_1, u, \cdots w_n) + \cdots$

$= a_1 f(u, w_2 \cdots w_n) - a_2 f(u, w_1, w_3 \cdots w_n)$

$+ \cdots$

$= a_1 f'(w_2 \cdots w_n) - a_2 f'(w_1, w_3 \cdots w_n)$

$+ a_3 \cdots$

$= 0$

T 满射. 对任意 $g: W \times \cdots \times W \to \mathbb{R}$

①. ②. ③.

定义 $f: V \times \cdots \times V \to \mathbb{R}$.

$f(v_1, \cdots v_n) = a_1 g(w_2, \cdots w_n) - a_2 g(w_1, w_3 \cdots w_n)$

$(v_i = a_i u + w_i)$ $+ a_3 g(w_1, w_2, w_4 \cdots w_n) \cdots$

验证 $f$. ①. ②. $\boxed{③}$ 练习.

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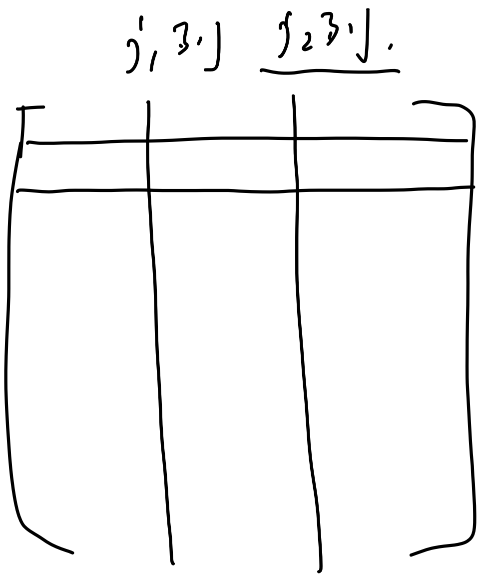
展开公式

$A = (a_{ij})_{n\times n}$.

$$|A| = \sum_{j_1=1}^{n} (-1)^{1+j_1} a_{1j_1} \cdot |M_{1j_1}|$$

$$= \sum_{j_1=1}^{n} (-1)^{1+j_1} a_{1j_1} \cdot \sum_{j_2 \neq j_1} a_{2j_2} \cdot |(M_{1j_1})_{2j_2}| (-1)^{?}$$

$(-1)^{?}$ $\quad |M_{12,\, j_1 j_2}|$




$\boxed{M_{K,L}.}$

$K \subset \{1 \cdots n\}$

$L \subset \{1 \cdots n\}$

$j_2 > j_1$ 时. 原 $j_2$ 列是 $M_{1j_1}$ 的 $j_2 - 1$ 列

$j_2 < j_1$ 时. 原 $j_2$ 列是 $M_{1j_1}$ 的 $j_2$ 列

$$? = \begin{cases} \underline{(-1)_2 - 1} \text{ 或者 } 2+j'_2 & j'_2 > j'_1 . \\ 2+j'_2+1 & j'_2 < j'_1 \end{cases}$$

$$= \sum_{j_1 \cdots j_n \text{ 互不相同.}} (-1)^{\underline{1+j_1+2+j'_2+\cdots+n+j_n} + \underline{l(j_1\cdots j_n)}}$$

$$\boxed{(-1)^{l(j_1\cdots j_n)} \underline{a_{1j_1}} \; a_{2j_2} \cdots a_{nj_n}}$$

$l(j_1\cdots j_n) =$ $j_1 \cdots j_n$ 的逆序对的个数

$\# \{ \underline{j_k < j_l ,} \ \underline{k > l} \}$.

$n!$ 个乘积, $(n-1)\cdot n!$ 运算.

( Leibnitz rule for derivative of det A )

<u>$n=2$</u> $\begin{vmatrix} a & b \\ c & d \end{vmatrix} = (-1)^0 ad + (-1)^{-1} bc = ad - bc$

$l(\underline{1,2})$ $l(2,1)$

$j_1 j_2$ $j_1 j_2$

$$\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = aei + (-1)^2 dhc + (-1)^2 gbf.$$

$\underbrace{(3,1,2)}$ $\underbrace{(2.3.1)}$

$(-1)^3 ceg - hfa - ibd.$

$\underbrace{(3.2,1)}_{3个}$

例: $X = \begin{bmatrix} A & B \\ C & D \end{bmatrix}_{2n\times 2n}$ $A.B.C.D\ n\times n.$

$\boxed{AC = CA}$

假设 A可逆. $|X| = \begin{vmatrix} A & B \\ 0 & D-CA^{-1}B \end{vmatrix}$

$= |A| \cdot |D - CA^{-1}B|$

$= |A(D-CA^{-1}B)|$

$= |AD - CB|$

不可逆怎么办:

微扰法：$A+\lambda I = A_\lambda$.

$|A_\lambda| = \lambda$ n次多项式

$\mathbb{F} = \mathbb{R}$ 时. $A_\lambda C = C A_\lambda$

除了至多 n 个 $\lambda$ 外 $A_\lambda$ 可逆.

$$\begin{vmatrix} A_\lambda & B \\ C & D \end{vmatrix} = |A_\lambda D - CB|$$

成立，除了有限个 $\lambda$.

$$\lim_{\lambda \to 0} \Rightarrow \begin{vmatrix} A & B \\ C & D \end{vmatrix} = |AD - CB|$$

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推广到一般 $\mathbb{K}$（域）$M_{n\times n}(\mathbb{K})$.

$A+\lambda I = A_\lambda$. 视为 $\mathbb{K}[\lambda]$ 上的矩阵

$\mathbb{F} = \mathbb{K}(\lambda) = \left\{ \frac{f(\lambda)}{g(\lambda)} \middle| f, g \text{ polynomials of } \lambda, g(\lambda) \neq 0 \right\}$

$\mathbb{F}$ 域

$|A_\lambda| \neq 0$,

$$\begin{vmatrix} A_\lambda & B \\ C & D \end{vmatrix} = |A_\lambda D - CB| \text{ 在 } \mathbb{K}(\lambda) \text{ 成立}$$

$\Rightarrow$ 在 $\mathbb{K}[\lambda]$ 中成立.

代入 $\lambda = 0$ （与乘法，加法交换）

$$\begin{vmatrix} A & B \\ C & D \end{vmatrix} = |AD - CB|$$

（尝试 $(I - \alpha\beta^T)^{-1}$ ）

$\alpha, \beta \in M_{n\times 1}$.

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定义：伴随 $A^* = (a'_{ij})_{n\times n}$.

$$a'_{ij} = (-1)^{i+j} |A_{ji}|$$

$$A^* A = A A^* = (\det A) \cdot I.$$

$\det A$ 的第一行展开

$$\underbrace{\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ & & & \\ & & & \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} |A_{11}| & -|A_{21}| \\ -|A_{12}| & +|A_{22}| \\ |A_{13}| & \vdots \\ \vdots & \vdots \\ (-1)^{n+1}|A_{1n}| & A_{2n} \end{bmatrix}}_{A^*} = \begin{bmatrix} \Delta & 0 \\ & \end{bmatrix}$$

$$\det\begin{bmatrix} a_{11}, a_{12} & \cdots & a_{1n} \\ a_{11}, a_{12} & \cdots & a_{1n} \\ \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots \end{bmatrix} = 0$$ 第二行展开.

$, \quad A A^* = (\det A) I$

$$\det A \neq 0 \Rightarrow A^{-1} = \boxed{\frac{1}{\det A} \cdot \underset{\sim}{A^*}},$$

## $\boxed{\text{Cramer rule}}$.

$Ax = b$. $\quad$ $A$ 可逆.

$$x = A^{-1}b \Rightarrow \underline{\frac{1}{\det A} A^* \cdot b}$$

$$x_i = \frac{\det(\quad b \quad)}{\det A} \leftarrow \underline{A \text{ 的第 } i \text{ 列换成 } b}$$

Ex: $\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \cdot \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}$

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \frac{1}{a_{11}a_{22} - a_{12}a_{21}} \cdot \begin{bmatrix} a_{22} & -a_{12} \\ -a_{21} & a_{11} \end{bmatrix} \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}$$

$$= \frac{1}{\det A} \cdot \begin{pmatrix} \underline{a_{22} \cdot b_1 - a_{12} \cdot b_2} \\ a_{11} b_2 - a_{21} \cdot b_1 \end{pmatrix} \quad \begin{vmatrix} b_1 & a_{12} \\ b_2 & a_{22} \end{vmatrix}.$$

$$\begin{vmatrix} a_{11} & b_1 \\ a_{21} & b_2 \end{vmatrix}$$

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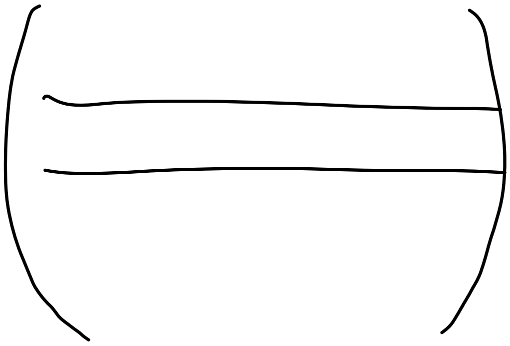
Laplace 公式 (行展开的推广)  $K \subset \{1 \dots n\}$ 子集.

$A =$ 

$$|A| = \sum_{\substack{L \subset \{1 \dots n\} \\ |L| = |K|}} (-1)^{\varepsilon_{K,L}} |M_{K,L}| \, |M_{K^c,L^c}|$$

$$\varepsilon_{K,L} = \Big(\sum_{i \in K} i\Big) + \Big(\sum_{j \in L} j\Big)$$

$V_1 = \operatorname{span}(e_i)_{i \in K}$. $V_2 = \operatorname{span}(e_i)_{i \in K^c}$.

$$V = V_1 \oplus V_2$$

$$f(v_1 \dots v_n) = f(v_1^1 + v_1^2,\ v_2^1 + v_2^2,\ \dots)$$

$$= \sum \underbrace{f(\qquad\qquad)}_{2^n \text{ 项}}$$

$$= \sum f(\ ,\ ,\ \ )$$

恰好 $|K|$ 列在 $V_1$ 中.

$n-|K|$ 列在 $V_2$ 中.

$$= \frac{\sum (-1)^{\square} f(\underbrace{\_\_\_}_{V_1}, \underbrace{\_\_\_}_{V_2})}{\underline{\det \square} \cdot \underline{\det \square}}$$

回忆:

<u>Cauchy-Binet</u> <u>$\det(AB)$</u> <u>$A_{m\times n}$.</u> <u>$B_{n\times m}$</u>

$$\det(AB) = \begin{cases} 0 & \text{if } m>n \\ \displaystyle\sum_{\substack{S\subset\{1\cdots n\} \\ |S|=m}} \det(A_{m,S}) \cdot \det(B_{S,m}) & \text{if } m\le n. \end{cases}$$

(How to prove, use properties of det)

Minor 与 rk 关系 $A \in M_{m\times n}(\mathbb{R})$

定理: $\max\{k \mid \underset{|M|\neq 0}{M}\ k\times k \text{ minor of } A\} = rk(A)$

证明: $|A_{K,L}| \neq 0 \iff A_{K,L}$ 可逆.

$rk(A_{K,L}) \le rk(A_{K,[n]}) \le rk\, A$.

$\Rightarrow$ 左边 $\le rk(A)$

若 $|A_{K,L}| \neq 0$, $\#K = \#L =$ 左边 $= k$

不妨假设 $K = \{1 \cdots k\}$. $L = \{1 \cdots k\}$

$A = (v_1 \cdots v_k, \cdots v_n)$

$v_1' \cdots v_k'$ 是 $v_1 \cdots v_k$ 的前 $k$ 行.

$M_{[k+1],[k+1]} = \begin{pmatrix} v_1' & \cdots & v_k' & v_{k+1}' \\ a_1 & \cdots & a_k & a_{k+1} \end{pmatrix}$ det $= 0$. $\Rightarrow$ 列不满秩 $\Rightarrow$ 线性相关.

$\Rightarrow \begin{pmatrix} v_{k+1}' \\ a_{k+1} \end{pmatrix} = c_1\begin{pmatrix} v_1' \\ a_1 \end{pmatrix} \cdots + c_k\begin{pmatrix} v_k' \\ a_k \end{pmatrix}$

且 $\begin{pmatrix} c_1 \\ \vdots \\ c_k \end{pmatrix} = (v_1' \cdots v_k')^{-1} \cdot v_{k+1}'$. 同理看前 $k+1$ 列. $k$ 行 + 后面行

$\Rightarrow \quad v_{k+1} = c_1 v_1 + \cdots + c_k v_k \ , \in \mathrm{Span}_{\mathbb{R}}(v_1, \ldots v_k)$

同理 $\Rightarrow \quad v_1 \cdots v_n \in \mathrm{Span}_{\mathbb{R}}(v_1, \ldots v_k)$

$\Rightarrow \quad rk(A) = k.$

目前为止，所有证明依赖于 $F$ 是域.

$\det: M_n(F) \to F$.

$A=(a_{ij}) \mapsto \sum (-1)^{\tau(j_1 \cdots j_n)} a_{1j_1} \cdots a_{nj_n}$

$j_1 \cdots j_n$ 互不相同.

已有等式

$$(*) \begin{cases} \det(AB) = \det A \det B \\ \det(A^T) = \det A \\ AA^* = A^*A = (\det A) I. \\ \det A \text{ 在行列变换下的性质.} \end{cases}$$

对一般环 $R$ 也成立.

方法一：直接验证

方法二：一种从 $F$ 到 $R$ 的方法.

取 $R_0 = \mathbb{Z}[x_{ij}, y_{ij}]$

$R_0 \hookrightarrow F_0 = \mathbb{Q}(x_{ij}, y_{ij})$ 域

$A_0 = (x_{ij})$. $B_0 = (y_{ij})$

则 $\det(A_0 B_0) \in R_0 \subset F_0$

$\det(A_0)\det(B_0) \in R_0 \subset F_0$

二者在 $F_0$ 中相等 $\Rightarrow$ 本身相等.

定义 $\varphi: R_0 \to R$

$x_{ij} \mapsto a_{ij}$

$y_{ij} \mapsto b_{ij}$

则 $\varphi(\det(A_0 B_0)) \overset{\text{为同态}}{=} \det\left((a_{ij})_{n\times n}(b_{ij})_{n\times n}\right)$

$\varphi(\det A_0 \cdot \det B_0) = \det(a_{ij})_{n\times n} \cdot \det(b_{ij})_{n\times n}$

所以有 $\det A \det B = \det(AB)$ 对 $A, B \in M_n(R)$ 成立.

其他类似.

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推论: $A \in GL_n(R)$, 当且仅当

$\det A$ 在 $R$ 中乘法可逆.

证明: $AB = I_n \Rightarrow \det A \cdot \det B = 1$

$\det A$ 可逆 $\Rightarrow \left((\det A)^{-1} A^*\right) A = A \cdot \left(\det A^{-1} A^*\right)$

$= I_n$

思考: 是否一定有左逆 = 右逆?!