RT Exam

August 11th 2025

If you use a conclusion from the homework, please also write down its proof on the exam paper.

Problem 1. *Consider the following elements (permutations) of the symmetric group* S_5*:*

$$\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 5 & 4 & 2 & 3 & 1 \end{pmatrix}, \quad \tau = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 3 & 4 & 5 & 1 & 2 \end{pmatrix}.$$

Compute the product $\sigma\tau$*.*

Problem 2. *Let* G *be a group of order* 15*. Is* G *always abelian? If so, prove it. If not, give a counterexample of a non-abelian group of order* 15*.*

Problem 3. *Write down the character table of the group* S_3*. Let* V *be the irreducible representation of* S_3 *with maximal dimension. Write down the irreducible decomposition of* $V \otimes V$ *(you only need to write down the multiplicity of each irreducible representation).*

Problem 4. *Find the number of Sylow* p*-subgroups in* $\mathrm{GL}(n, \mathbb{F}_p)$ *for a prime* p *and a positive integer* n*.*

Problem 5. *Let* G *be a group and let* H *be a subgroup of* G*. For a representation* $\varphi\colon G \to \mathrm{GL}(V)$*, the restriction to* H*, denoted* φ_H*, is the group representation defined by* $\varphi_H(h) = \varphi(h)$ *for all* $h \in H$*. Let* ρ *be the regular representation of* G *on* $\mathbb{C}[G]$*. Prove that the restriction of* ρ *to* H *is isomorphic to a direct sum of copies of the regular representation of* H*.*

Problem 6. *We call a group* G *simple if it is not the trivial group* $\{e\}$ *and its only normal subgroups are* $\{e\}$ *and* G*. Assume* G *is a non-abelian simple group. Prove that* G *has no faithful complex representation of dimension* 2*.*