RT HW3
Due 8/4 please submit your solutions to the TAs in tutorial session

July 29, 2025

Problem 1. *Classify subgroups and normal subgroups of D_n for $n \geq 3$.*

Problem 2. *Prove that $G = GL(2, \mathbb{F}_2)$ is isomorphic to S_3 by the action of G on $(\mathbb{F}_2)^2$.*

Similarly,define $PGL(2, \mathbb{F}_3) = GL(2, \mathbb{F}_3)/D$, where $D = \{\lambda I_2 \mid \lambda \in \mathbb{F}_3^\times\}$. Prove that $PGL(2, \mathbb{F}_3) \cong S_4$. Hint: Consider the action of $GL(2, \mathbb{F}_3)$ on the set of all one-dimensional subspaces of $(\mathbb{F}_3)^2$.

Problem 3. ***Second Isomorphism Theorem**. Let H be a normal subgroup of group G and K be a subgroup of G. Prove*

1. *$HK = \{hk | h \in H, k \in K\}$ is a subgroup of G.*
2. *$H \cap K$ is a normal subgroup of K.*
3. *There is an isomorphism $HK/H \cong K/H \cap K$.*

Problem 4. *Let $O_1, \cdots, O_k$ be all the conjugacy classes in a finite group G. Choose $x_i \in O_i$ and let $C_i = \{g \in G | g x_i g^{-1} = x_i\}$ (which is called the centralizer of x_i). Denote $n_i = |C_i|$. Prove*

$$\frac{1}{n_1} + \frac{1}{n_2} + \cdots \frac{1}{n_k} = 1$$

Problem 5. *(Semidirect product) Let H and K be two groups and $\phi\colon K \to$ $\mathrm{Aut}(H)$ be a group homomorphism. Define a binary operation on $H \times K$ by $(h, k)(h', k') = (h\varphi(k)(h'), kk')$. Check this binary operation gives a group structure. Prove that the subsets $\{e_H\} \times K$ and $H \times \{e_K\}$ are subgroups of this group and $H \times \{e_K\}$ is a normal subgroup.*

Problem 6. *Let G be a group generated by real valued functions $f = \frac{1}{x}$ and $g = \frac{x-1}{x}$ via composition of functions. Prove that G is isomorphic to S_3.*

Problem 7. *Find the number of Sylow p-subgroups in $\mathrm{GL}(n, \mathbb{F}_p)$ for a prime p and a positive integer n.*

Problem 8. *Find the number of isomorphism classes of groups of order 55.*