RT HW4
Due 8/6, please submit your solutions to the TAs in tutorial session

August 2, 2025

Problem 1. *Let $\mathbb{Z}/n\mathbb{Z}$ be the residue classes modulo positive integer n. Assume $(\mathbb{Z}/nZ)^\times$ is the set of residue classes coprime to n. It forms a group under usual multiplication of residue classes. Prove that the group of automorphisms of $\mathbb{Z}/n\mathbb{Z}$ is isomorphic to the group of units $(\mathbb{Z}/n\mathbb{Z})^\times$.*

Problem 2. *In this exercise, we will investigate the conjugacy classes of the symmetric group S_n. Let $i_1, \cdots, i_l$ be distinct integers in $\{1, \cdots, n\}$. A cycle $(i_1, \cdots, i_l)$ is a permutation of S_n that sends i_1 to i_2, i_2 to i_3, ..., i_{l-1} to i_l, i_l back to i_1, and leaves other integers fixed. The number l is called the length of the cycle. Two cycles $(i_1, \cdots, i_l)$ and $(j_1, \cdots, j_k)$ are disjoint if they do not share any common integers.*

1. *Prove that two disjoint cycles commute, i.e., if $(i_1, \cdots, i_l)$ and $(j_1, \cdots, j_k)$ are disjoint cycles in S_n, then*
$$(i_1, \cdots, i_l)(j_1, \cdots, j_k) = (j_1, \cdots, j_k)(i_1, \cdots, i_l).$$
2. *Prove that every permutation in S_n can be written as a product of disjoint cycles. This product is unique up to the order of the cycles. (Hint: consider the action of $\langle\sigma\rangle$ on $[n]$ and the orbits.)*
3. *Let $\sigma = (i_1, \cdots, i_l)$ be a cycle of length l in S_n. Prove that $\tau\sigma\tau^{-1} = (j_1, \cdots, j_l)$ is a cycle of length l for any $\tau \in S_n$, where $j_k = \tau(i_k)$.*
4. *Prove that the conjugacy classes of S_n are in one-to-one correspondence with the partitions of n.*

Problem 3. *Write $O(2)$ as a semi-direct product of $SO(2)$ with $\mathbb{Z}/2\mathbb{Z}$, and $O(3)$ as a direct product of $SO(3)$ with $\mathbb{Z}/2\mathbb{Z}$.*

Problem 4. *A rigid motion of the plane is a map $f\colon \mathbb{R}^2 \to \mathbb{R}^2$ of the form*
$$f(x) = Ax + b$$

where A is a 2×2 orthogonal matrix, i.e., $A \in O(2)$, and $b \in \mathbb{R}^2$ is a vector in the plane. The rigid motions of the plane form a group G under composition of maps. Prove that the group G preserves the distance between points in $\mathbb{R}^2$, i.e., for any $f \in G$ and any points x and y in $\mathbb{R}^2$, we have

$$d(f(x), f(y)) = d(x, y),$$

where d is the usual Euclidean distance in $\mathbb{R}^2$. The group G of rigid motions of the plane is isomorphic to the semi-direct product $\mathbb{R}^2 \rtimes O(2)$.

Problem 5. *Find the number of conjugacy classes and the number of elements in each conjugacy class for the rotation symmetry groups of Platonic solids. You can use the fact that $T \cong A_4$, $O \cong S_4$ and $I \cong A_5$. Can you interpret your result in terms of the geometry of Platonic solids? (Hint: use the geometric interpretation of BAB^{-1} for $A \in SO(3)$ and $B \in SO(3)$.)*

Problem 6. *Let P_1, P_2 be two planes in $\mathbb{R}^3$ intersecting at a line l. Show that the composition of the two reflections with respect to P_1 and P_2 is a rotation around l. The rotation angle is twice the angle between the two planes.*