RT HW5
Due 8/12, please submit your solutions to the TAs in tutorial session

August 5, 2025

Problem 1. *Let G_1 and G_2 be two groups and V_i is an representation of G_i over $\mathbb{C}$ for $i = 1, 2$. Define the representation $V_1 \otimes V_2$ of $G_1 \times G_2$ by*

$$(g_1, g_2) \cdot (v_1 \otimes v_2) = (g_1 \cdot v_1) \otimes (g_2 \cdot v_2)$$

for $g_i \in G_i$ and $v_i \in V_i$. Prove that $V_1 \otimes V_2$ is irreducible if and only if both V_1 and V_2 are irreducible. How do you express the character of $V_1 \otimes V_2$ in terms of the characters of V_1 and V_2?

Problem 2. *Let V be an irreducible representation of a finite group G over $\mathbb{C}$. Assume V is not the trivial representation. Prove that for any $v \in V$, we have $\sum_{g \in G} g \cdot v = 0$. (For cyclic groups, this is known to be an identity of roots of unity.)*

Problem 3. *Let G be a finite group and operates on a finite set X. Let V be the vector space over $\mathbb{C}$ with basis X, i.e., $V = \mathbb{C}^X$. Then the G-action on X induces a linear representation of G on V. Show that number of orbits of G on X is equal to multiplicity of the trivial representation in the irreducible decomposition of V.*

Problem 4. *Prove the following combinatorial identity by characters*

$$\sum_{\sigma \in S_n} (\text{number of elements in } [n] \text{ fixed by } \sigma)^2 = 2(n!)^2$$

1. *Let V be the representation of S_n on $\mathbb{C}^n$ induced by the action of S_n on standard basis $e_1, \cdots, e_n$ of $\mathbb{C}^n$. Show that the character χ_V of V is given by*

$$\chi_V(\sigma) = \text{number of elements in } [n] \text{ fixed by } \sigma$$

   for $\sigma \in S_n$.

2. *In class, we showed that the trivial representation of S_n is contained in the representation V given by subspace spanned by $e_1 + e_2 + \cdots + e_n$. The*

G-invariant compliment W is defined by $W = \{v = (v_1 \cdots v_n)^T \in \mathbb{C}^n \mid \sum_{i=1}^n v_i = 0\}$. Show that W is irreducible by the following method. Let V' be a nonzero G-invariant subspace of W and $v \in V'$ be a non-zero vector. If any two component v_i and v_j of v are not equal, then use a permutation $\sigma \in S_n$ to prove that $e_i - e_j$ is also in V'. Then prove that V' must be all of W.

3. *Use character of V to prove the combinatorial identity stated in the beginning.*