RT HW6

For you to practice representation theory, you can submit to the TAs if you want to get feedback.

August 8, 2025

Problem 1. *Let* $[G, G]$ *be subgroup of* G *generated by all commutators* $[g, h] = ghg^{-1}h^{-1}$ *for* $g, h \in G$.

1. *Show that* $[G, G]$ *is a normal subgroup of* G.
2. *Show that* $G/[G, G]$ *is abelian.*
3. *Show that if* H *is abelian and* $\phi \colon G \to H$ *is a group homomorphism, then* $[G, G] \subset \ker \phi$.
4. *Prove that there is an one-to-one correspondence between the one-dimensional representations of* G *and the irreducible representations of* $G/[G, G]$.

Problem 2. *Let* $\mathrm{SL}(2, \mathbb{R})$ *be the group of* 2×2 *real matrices with determinant* 1 *and* W *be the real vector space of* 2×2 *real matrices with trace* 0.

1. *Show that* W *is a real vector space of dimension* 3.
2. *Show that* W *is a* $\mathrm{SL}(2, \mathbb{R})$*-representation by defining the action of* $\mathrm{SL}(2, \mathbb{R})$ *on* W *by*
$$A \cdot X = AXA^{-1}$$
for $A \in \mathrm{SL}(2, \mathbb{R})$ *and* $X \in W$.
3. *Show that the bilinear form on* W
$$\langle X, Y \rangle = \mathrm{tr}(XY)$$
is $\mathrm{SL}(2, \mathbb{R})$*-invariant.*
4. *Find the signature* (p, q) *of the bilinear form* $\langle \cdot, \cdot \rangle$.
5. *(Optional) Let* $O(p, q)$ *be the group of linear transformations on* W *that preserve the bilinear form* $\langle \cdot, \cdot \rangle$. *Find the image and kernel of the representation* $\mathrm{SL}(2, \mathbb{R}) \to O(p, q)$.

Problem 3. *From class, we know that the character* χ_{reg} *of regular representation satisfies* $\chi_{reg}(g) = 0$ *if* $g \neq e$. *There is an inverse of this proposition. Let* χ *be a character of* G *and satisfies* $\chi(g) = 0$ *if* $g \neq e$. *Prove that the corresponding representation is the direct sum of several copies of regular representation, i.e.* $\rho \cong \rho_{reg} \oplus \rho_{reg} \cdots \oplus \rho_{reg}$ *for some integer* n.