Representation theory projects

1 Project 1 Groups: generators and relations

In this project, we will study groups generated by reflections.

1.1 Symmetric group

Let S_n be the symmetric group on n elements. In homework, you studied the generators of S_n as transpositions, $s_i = (i, i+1)$ and their relations. More precisely,

1. Show that every element in S_n can be written as a product of elements in $\{s_1, s_2, \ldots, s_{n-1}\}$.
2. Prove that the elements satisfy the following equalities:

$$\begin{aligned} s_i s_j &= s_j s_i \quad \text{if } |i-j| > 1, \\ s_i s_{i+1} s_i &= s_{i+1} s_i s_{i+1}, \\ s_i^2 &= e \quad \text{for all } i. \end{aligned}$$

3. For each permutation $\sigma \in S_n$, define the number of inversions of σ as the number of pairs (i, j) such that $i < j$ and $\sigma(i) > \sigma(j)$. Show that the number of inversions is equal to the minimal number of s_i used to express σ as their product (counted with multiplicities). For example, the number for $\sigma = s_1 s_2 s_1$ is three.
4. Show that the relations above are sufficient to determine the group S_n, i.e., for any two different expressions of the same element in S_n, they can be transformed into each other using the relations above.
5. Find the minimal sets of transpositions $s_{ij} = (i, j)$ that generate S_n. How many are there?
6. For each set of such generators, find the relations to determine the group S_n and prove your conclusion.

1.2 Dihedral group

Let D_n be the symmetry group of a regular n-gon.

1. Find the minimal sets of reflections that generate D_n.
2. How many such sets are there?
3. Find the relations for each set of generators and prove your conclusion.

1.3 Symmetry group of Platonic solids

Let G be the symmetry group of a Platonic solid.

1. Find the minimal sets of reflections that generate G.

2. How many such sets are there?

3. Find the relations for one set of generators and prove your conclusion.

1.4 Symmetry group of tiling patterns

Let T be the tiling of the plane by equilateral triangles, and G be the symmetry group of such a pattern.

1. Find one set of reflections that generate G.

2. Find the relations for the generators you choose and prove your conclusion.

3. Can you generalize your result to other regular tiling patterns, such as square or hexagonal tiling?

4. Choose one set of reflections that generate G. For each element σ, find the number of reflections used to express σ as their product.

2 Project 2 Groups and bilinear forms

In this project, we will study groups arising from certain shapes of hexagons. Consider convex hexagons P with inner angles $\frac{2\pi}{3}$. Let the lengths of the sides be $a_1, \cdots, a_6$ in the counterclockwise orientation, and form a vector $a = (a_1, \cdots, a_6) \in \mathbb{R}^6$.

1. Find the vector space W spanned by all such a.

2. Consider the group G generated by linear transformations $L_i, \quad i = 1, \cdots, 6$ of W in the form of

   $$L_i \colon a_i \mapsto -a_i, a_{i-1} \mapsto a_{i-1} + 2a_i, a_{i+1} \mapsto a_{i+1} + 2a_i, a_j \mapsto a_j, \text{for other } j$$

   Check that W is invariant under the action of G and that G is an infinite group. Here indices are taken modulo 6.

3. Find a bilinear form $B \colon W \times W \to \mathbb{R}$ that group G preserves.

4. Is this bilinear form unique? If not, find all bilinear forms that are preserved by the group G.

5. Denote by F the set of vectors a from all such hexagons. Fix a bilinear form in the previous questions such that $B(a, a) > 0$ for some $a \in F$. Let C be the vectors v in W such that $B(v, v) > 0$. Show that F is a subset of C. Describe the union of orbits of F under the action of G.

6. Find finite subgroups of G. Can you find a classification of these kinds of finite subgroups?

7. Find the relations for the generators L_i and prove your conclusion. (Hard problem, you may just try to find and guess the relations, proving that they are all the relations is a hard problem.)

8. Consider the shapes of P such that it admits a tiling (decomposition) into regular triangles with unit length. Can you describe the possible vectors a from such P and their G-orbits in W? What are the possible numbers of triangles used in the tiling and for each n, is there a counting formula for $c(n)$ the number of different shapes of hexagons with n triangles? (Hard problem, but you can try to find a few examples and reduce to an arithmetic problem, other shapes may result in a simpler form.)

9. Try the problem with other shapes, for example a quadrilateral, a pentagon with certain inner angles you prefer.

3 Project 3 McKay conjecture

In this project, we will verify the McKay conjectures for some groups. Let G be a finite group and P is a Sylow p-subgroup of G. Denote by $N_G(P)$ the normalizer of P in G. The McKay conjecture states that the number of irreducible representations of G with dimension coprime to p is equal to the number of irreducible representations of $N_G(P)$ with dimension coprime to p. It was recently proved after a series of works by many mathematicians.

3.1 Finite subgroups of $SO(3)$ and $SU(2)$

1. Consider the dihedral group D_n. List all the irreducible representations of D_n and their dimensions.

2. Give the classification of Sylow p-subgroups of D_n and their normalizers.

3. For $p = 2$, verify the McKay conjecture for D_n.

4. For general prime number p, verify the McKay conjecture for D_n.

5. Consider finite subgroups of $SO(3)$ and $SU(2)$. List all the irreducible representations of these groups and their dimensions.

6. Verify the McKay conjecture for finite subgroups of $SO(3)$ and $SU(2)$.

3.2 Groups of order pq and p^2q

Assume p and q are two distinct prime numbers.

1. Let G be a group of order pq. List the possible isomorphism classes of G.
2. For each isomorphism class of G, list all the irreducible representations of G and their dimensions.
3. Give the classification of Sylow p-subgroups of G and their normalizers.
4. Verify the McKay conjecture for G.
5. Generalize to groups of order p^2q.

3.3 Groups of other orders

Can you generalize the McKay conjecture to groups of other types discussed in class? For example, for $\mathrm{GL}(2, \mathbb{F}_p)$ and its Sylow p-subgroups.

4 Project 4 McKay graph for finite groups

In this project, we will explore the McKay graph for other finite groups not discussed in class. Let V be a fixed representation of finite group G. Consider all the isomorphism classes of irreducible representations V_i of G. View all V_i as the vertices, and if V_j appears in the irreducible decomposition of $V \otimes V_i$ with multiplicity n_{ij}, then draw n_{ij} edges from V_i to V_j. The resulting graph is called the McKay graph of G with respect to V.

4.1 Symmetric group S_n

1. Consider S_n and the standard representation V of S_n on $\mathbb{C}^n$. Construct all irreducible representations of S_3 and S_4.
2. Draw the McKay graph for S_3 and S_4 with respect to the standard representation V.
3. How does the McKay graph change when we consider other representations V of S_3, S_4?

4.2 Subgroups in $SO(3)$

1. Consider all finite subgroups in $SO(3)$. Construct all irreducible representations of G.
2. Let V be the standard representation of $SO(3)$ on $\mathbb{R}^3$ and view it as a representation on $\mathbb{C}^3$. Draw the McKay graph for each finite subgroup in $SO(3)$ with respect to the standard representation V.
3. How does the McKay graph change when we consider other representations V of G?

4.3 Finite subgroups in $U(2)$

1. Classify all finite subgroups in $U(2)$. Construct all irreducible representations of G.
2. Let V be the standard representation of $U(2)$ on $\mathbb{C}^2$. Draw the McKay graph for each finite subgroup in $U(2)$ with respect to the standard representation V.
3. How does the McKay graph change when we consider other representations V of G?

4.4 Other finite groups

You can also try a similar problem for finite groups of order pq or p^2q for distinct primes p and q. Make your own choice of V.