Representation Theory of groups

Three parts

① Groups and group operations

② Representations of (finite) groups character theory.

③ McKay correspondence and root system.

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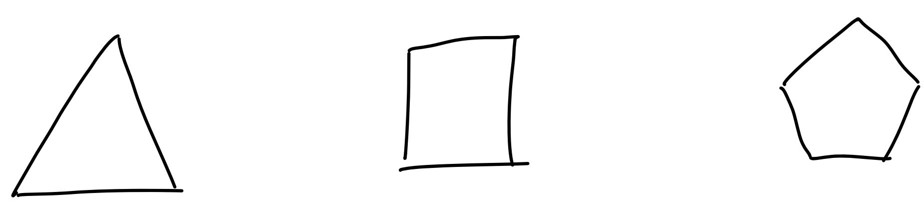
① Groups.

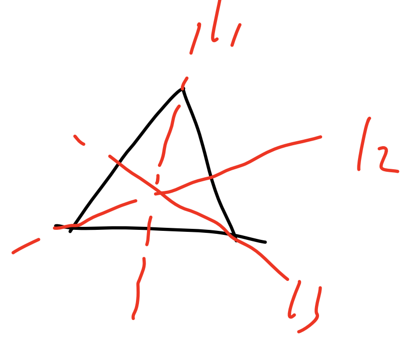
Group theory is a way to describe the symmetry of objects.

Ex: Symmetry of geometric objects.

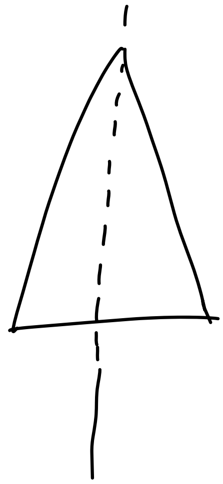
regular n-gon



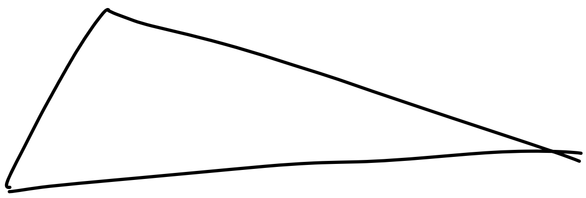
rotations by $0, 120°, 240°$, reflection by l_1, l_2, l_3.



Isosceles triangle



two symmetries.



"general" triangle.

(Not special)

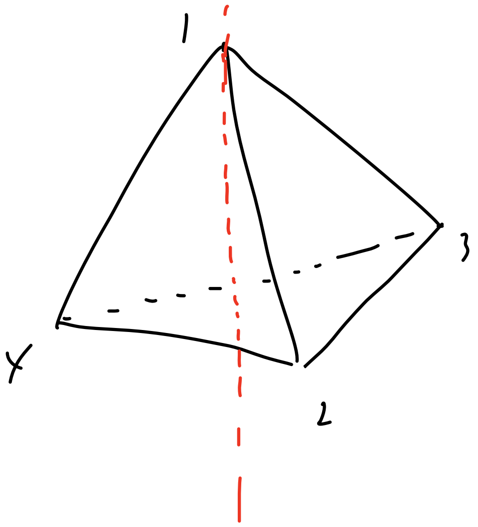
one symmetry

# of symmetries reflects the information of shapes.

"More symmetries" means more special.

Regular n-gons are the most special n-gons

Higher dim'l



Tetrahedron

Count # of rotations

$$\underline{1} + \underline{4 \times 2} + \underline{3 \times 1} = 12$$

0° $\quad 120^\circ$ $\quad 90^\circ$

$0,\quad \frac{\pi}{3}, \quad \frac{\pi}{4}$

# of reflections = 6.

Other elements

"6"

rotations composed with reflections

Another way to count

symmetries are classified by 4 different kinds by looking at where 1 is moved to.

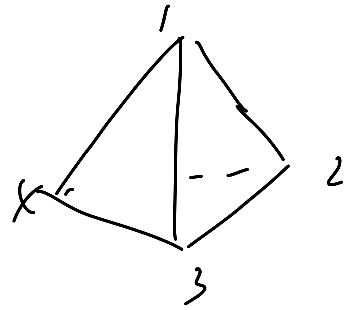
$1 \longmapsto 1$

$1 \longmapsto 2$

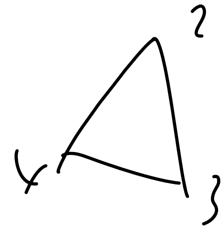
$1 \longmapsto 3$

$1 \longmapsto 4$

If $1 \mapsto 1$ then,



symmetry of



$\# = 6$

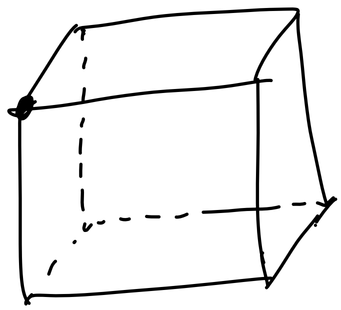
If $1 \mapsto 2$. Choose one symmetry σ, and use σ^{-1} to move 2 back to 1. then any τ of this kind $\sigma^{-1}\tau$ is of the first kind

$\Rightarrow$ $\#$ of such $\sigma^{-1}\tau = 6$

$= \#$ of such τ.

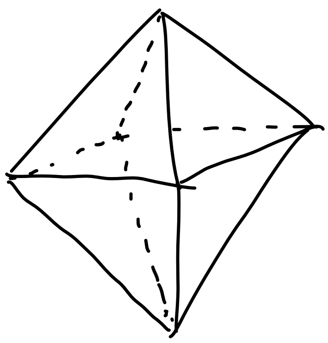
For the same reason,

$\# = 6 \times 4 = 24$



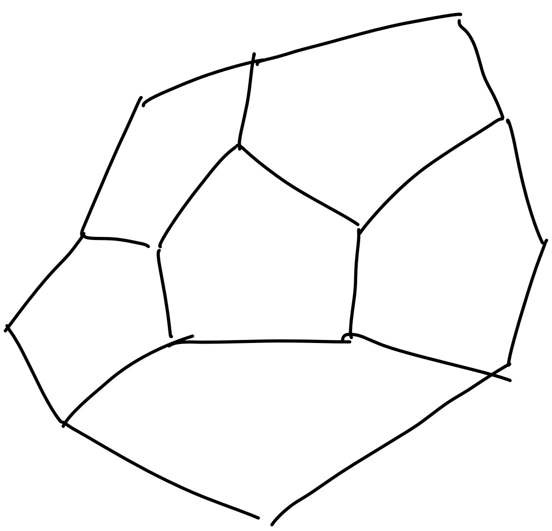
Cube $\# = 8 \times 6 = 48$

in pairs

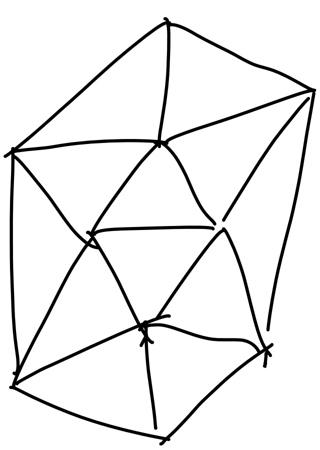


Octahedron $\# = 6 \times 8 = 48$

Dodecahedron



Twelve faces. each is a pentagon $\# = 12 \times 10 = 120$



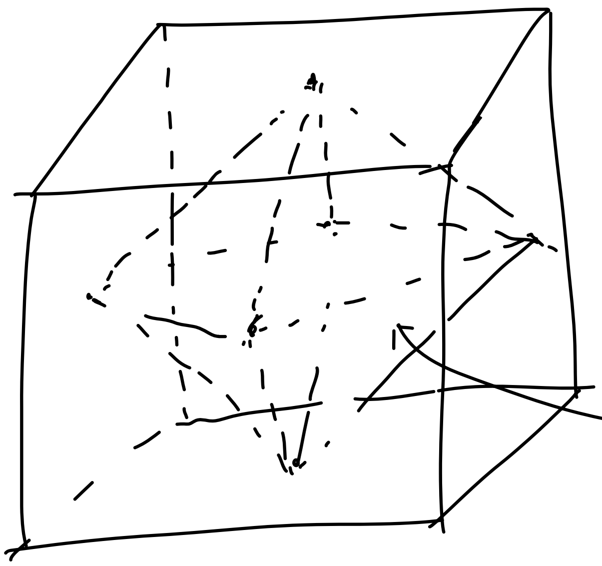
Icosahedron

in pairs

20 faces

$\# = 20 \times 6 = 120$

not just # of symmetries



also the symmetries can be identified.

Octahedron

"groups are isomorphic"

Ex: Symmetry of equations

$x^2+1=0$. $x^2-2=0$

$x^2+x+1=0$

$a+b\sqrt{-1}$. $a, b \in \mathbb{Q}$.

$\downarrow$

$a-b\sqrt{-1}$.

preserving the structure of algebra "+," "×"

$a + b\sqrt{2} \longmapsto a - b\sqrt{2}$. "the same" symmetries

$x^2 + x + 1 = 0 \Rightarrow x = \frac{-1 \pm \sqrt{-3}}{2}$

$\sqrt{-3} \longmapsto -\sqrt{-3}$. "the same" symmetries

When two equations have the same symmetries, the methods to solve them are the same.

"order 2" $\Leftrightarrow$ square root

Gauss, Galois $\Longrightarrow$ solves the problem of "ruler and compass"

$x^3 + ax^2 + bx + c = 0.$ how to solve.

" $\sqrt[3]{\ }$ ", " $\sqrt{\ }$ "

x^4 . . . —

" $\sqrt[4]{\ }$ ", " $\sqrt[3]{\ }$ ", " $\sqrt{\ }$ "

$x^5 + \dots$ — — Not solvable by radicals

$\Uparrow$ Abel-Ruffini, Galois.

symmetries determine solvability.

Ex: Symmetry of Physics laws

More complicated symmetry

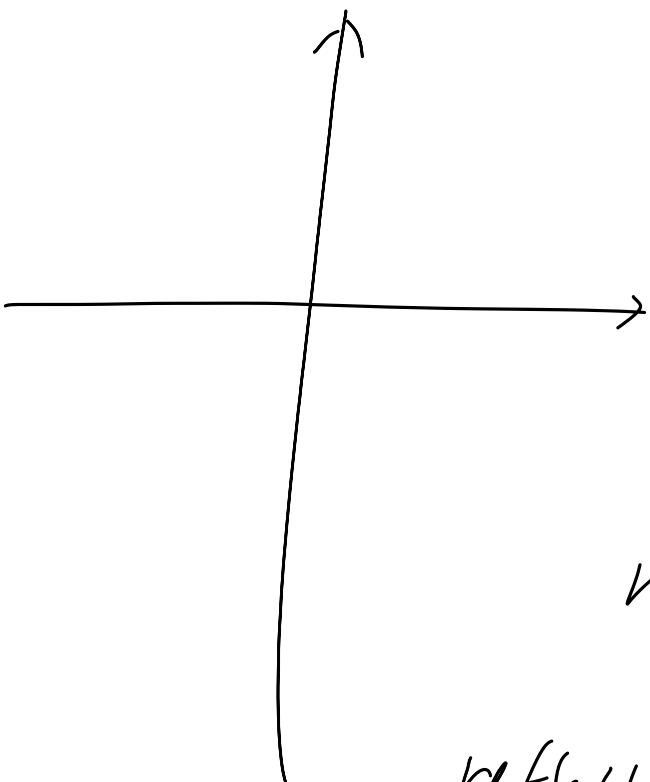
# of symmetry of physics laws (theories or models) is usually infinite

1D space "dim" of symmetries $= 1$

Translation + reflection

$x \mapsto x + a$ $\quad$ $y \mapsto b - y$

2D



translation

$$\begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} a \\ b \end{pmatrix}$$

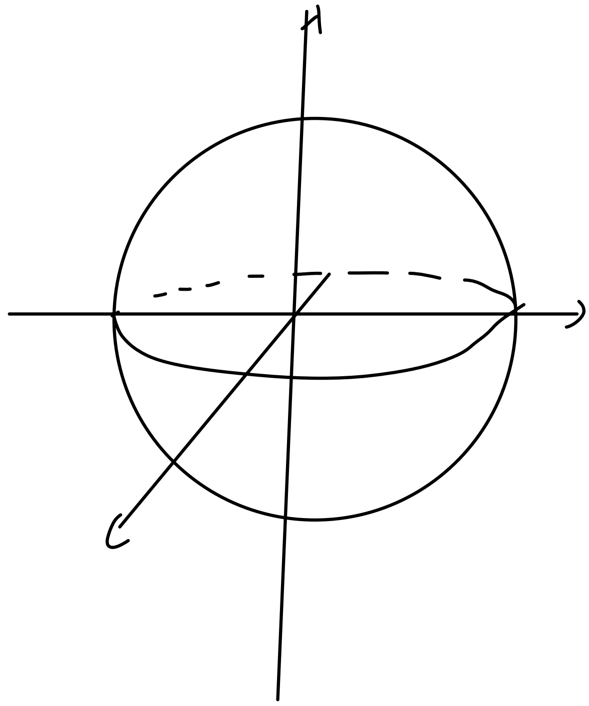
rotation. by angle θ

reflection, glide reflection

dim $= 2 + 1 = 3$.

3D translation dim = 3

rotation dim = 3



3 + 3

Time translation symmetry

Noether $\Longrightarrow$

Time translation $\Rightarrow$ energy

Translation $\Longrightarrow$ momentum

Rotation $\Longrightarrow$ angular momentum.

Electric-Magnetic theory. "symmetry of wave function"

$\Rightarrow$ electric charge.

"Modern Physics" Look for symmetries.

$\Downarrow$

conservation laws.

Algebra : a way to describe

operations. structures.

Algebraic approach to symmetries.

Group Theory

Defn : (Binary operation) X set.

$$* : X \times X \to X$$

$$(a, b) \mapsto a * b$$

Defn: (Group) A group G is a set with binary operation "$\cdot$" called multiplication (or product)

$\cdot : G \times G \to G$.

$(a, b) \mapsto a \cdot b$

(We usually drop "$\cdot$", $a \cdot b = ab$)

The binary operation satisfies

① Associativity $(ab)c = a(bc)$

② Identity element $\exists e \in G$, s.t. $\forall a \in G$, $ea = ae = a$

(Later we prove such e is unique)

③ Inverse element $\forall a \in G$, $\exists b \in G$. s.t. $ab = ba = e$.

(Later we prove such b is determined by a, and call it a^{-1})

Prop: (Uniqueness of identity element)

If e_1, e_2 are two identity elements, then $e_1 = e_2$

Pf: $e_1 e_2 = e_1$. (by e_2 being identity element)

$e_1 e_2 = e_2$ (by e_1 being identity element)

So $e_1 = e_2$ $\square$.

Prop: (Uniqueness of inverse element)

(How to phrase such a theorem)?? Stating what you want to prove is sometimes more important than proving it)

Fix $a \in G$, if $b, c \in G$ are inverses of a, in other words,

$$ba = ab = e, \qquad ca = ac = e.$$

then $b = c$.

Pf: Consider $(ba)c = b(ac)$

$$(ba)c = e \cdot c = c$$

$$b(ac) = b \cdot e = b$$

$$\Rightarrow \quad b = c \qquad \square$$

Ex:

$(\mathbb{Z}, +)$

$(\mathbb{Z}/n\mathbb{Z}, +)$ residue classes $\{\bar{0}, \bar{1}, \dots, \overline{n-1}\}$ modulo n.

$(\mathbb{Q}, +)$

$(\mathbb{Q}^\times = \mathbb{Q} \setminus \{0\}, \times)$

p prime number, $\mathbb{F}_p = \mathbb{Z}/p\mathbb{Z}$. $\quad (\mathbb{F}_p^\times = \mathbb{F}_p \setminus \{0\}, \times)$.

Non-Ex : (Odd numbers, +)

$(\mathbb{Z}_{\geq 0}, +)$

$(\mathbb{R}^3$, cross product$)$

Try to find examples satisfying 2. 3, but not 1.

Notation: $a_1 a_2 a_3 \dots a_n = ((a_1 a_2) a_3) \cdots a_n$

or any other form $= a_1((a_2 a_3) a_4) \cdots$

Associativity $\Rightarrow$ No ambiguity

or "well-defined"

These groups are special because the products are commutative $ab = ba$.

This not always the case.

Symmetric group (Not symmetry group)

Defn: $[n] = \{1, 2 \cdots n\}$

Defn: $S_n = \{ \text{bijections } \sigma : [n] \to [n] \}$.

$\sigma \cdot \tau =$ composition $\sigma \circ \tau$.

In other words:

$$\sigma \cdot \tau (i) = \sigma(\tau(i))$$

Prop: $(S_n, \cdot)$ is a group.

Before the proof, we introduce some notation.

An element σ in S_n is also called a permutation. and can be written as

$$\sigma(1), \cdots \sigma(n), \quad \text{or} \quad \begin{pmatrix} 1 & \cdots & n \\ \sigma(1) & \cdots & \sigma(n) \end{pmatrix}$$

For example, $n=3$. a permutation $1, 3, 2$

$$1 \mapsto 1, \quad 2 \mapsto 3, \quad 3 \mapsto 2. \qquad \text{or} \qquad \begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix}$$

Permutations are exactly all the maps $\{1, 2 \dots n\}$ to itself that are one-to-one (bijective). $f(i) = f(j)$ implies $i = j$. injective

f is also surjective, for any j. $\exists i$, such that $f(i) = j$

Pf: (1) Associativity. $(f \circ g) \circ h = f \circ (g \circ h)$.

$$(f \circ g) \circ h \, (i) = (f \circ g)(h(i))$$
$$= f(g(h(i)))$$

$$f \circ (g \circ h) \, (i) = f((g \circ h)(i))$$
$$= f(g(h(i)))$$

$$\Rightarrow (f \circ g) \circ h = f \circ (g \circ h)$$

(2) Identity element.

e is the identity map $\{1, 2 \dots n\}$ to itself.

$e(i) = i$.

$e \circ f(i) = e(f(i)) = f(i) \Rightarrow e \circ f = f$

$f \circ e(i) = f(e(i)) = f(i) \Rightarrow f \circ e = f$.

(3) Inverse: $f : \{1, 2 \dots n\} \to \{1, 2 \dots n\}$

f bijective.

Define $f^{-1} : \{1, 2 \dots n\} \to \{1, 2 \dots n\}$

$f^{-1}(i)$ is the unique element j such that $f(j) = i$.

Check $f \circ f^{-1} = \underline{f^{-1} \circ f = e}$.

from the definition. $f^{-1}(i) = j$ if $f(j) = i$.

$$f(f^{-1}(i)) = i.$$

$$\Rightarrow f\circ f^{-1}(i) = i. \quad \Rightarrow \underline{f\circ f^{-1} = e}$$

Compute $\quad f\circ(f^{-1}\circ f) = (f\circ f^{-1})\circ f$

$$= e\circ f = f.$$

$$\Rightarrow \underline{f}(f^{-1}\circ f(i)) = \underline{f}(i)$$

f is injective. $\Rightarrow f^{-1}\circ f(i) = i.$

$$\Rightarrow \quad f^{-1}\circ f = e.$$

$\square$.

Computation: It is convinient to write a permutation in 2 lines:

$$(i_1, i_2, \cdots i_n) \longleftrightarrow \begin{pmatrix} 1 & 2 & 3 & \cdots & n \\ i_1 & i_2 & i_3 & \cdots & i_n \end{pmatrix}$$

If $\quad f = \begin{pmatrix} 1 & 2 & \cdots & n \\ i_1 & i_2 & i_3 \cdot & i_n \end{pmatrix}$

$$g = \begin{pmatrix} 1 & 2 & & n \\ j_1 & j_2 & \cdots & j_n \end{pmatrix}$$

To compute $g \circ f$, we permute the columns of g such that the top line is $i_1\, i_2 \cdots i_n$

$f = (1, 3, 2, 4)$

$g = (2, 1, 3, 4)$

$$f = \begin{pmatrix} 1 & 2 & 3 & 4 \\ \downarrow & \downarrow & \downarrow & \downarrow \\ 1 & 3 & 2 & 4 \end{pmatrix} \quad g = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 3 & 4 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 3 & 2 & 4 \\ & \downarrow & \downarrow & \downarrow \\ 2 & 3 & 1 & 4 \end{pmatrix}$$

$$g \circ f = \underline{\begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 1 & 4 \end{pmatrix}}$$

To compute $f \circ g$. $f = \begin{pmatrix} 2 & 1 & 3 & 4 \\ 3 & 1 & 2 & 4 \end{pmatrix}$

$$f \circ g = \underline{\begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 1 & 2 & 4 \end{pmatrix}}$$

$g \circ f \neq f \circ g$.

S_4 is not commutative under composition.