Group representations: $G$ group.

$GL(n, F) = \{A \in M_n(F) \mid \det F \neq 0\}$. $F$ field. $\mathbb{C}, \mathbb{R}, \mathbb{F}_p$

$V$ $F$-linear space, $GL(V) = \{f : V \to V \mid f$ $F$- $\}$

$\dim V = n$, $\{e_1, e_2, \dots e_n\} = B$

$R_B(f)$ $f(e_1, e_2, \dots e_n) = (e_1 \dots e_n) \cdot R(f)$

$B' = (v_1 \cdots v_n) = (e_1 \cdots e_n) \cdot P$.
another basis $\Rightarrow P \in GL(n, F)$

Then $f(v_1 \cdots v_n) = f(e_1 \dots e_n) \cdot P$
$= (e_1 \cdots e_n) R_B(f)$
$= (v_1 \cdots v_n) P^{-1} R_B(f) P$

$$R_{B'} = P^{-1} R_B P$$

① Matrix rep'n. $R : G \to GL(n, F)$ group homomorphism

② linear rep'n on vector space $V$: $\rho : G \to GL(V)$ group homo

③ linear operation: $G \times V \to V$.

$$(g, v) \mapsto g \cdot v$$

$$g(v+w) = gv + gw, \quad g(\lambda v) = \lambda gv,$$

① ⇒ ②. $G \to GL(n, F) \xrightarrow{\simeq} GL(F^n)$
$A \longmapsto (v \mapsto Av)$

② ⇒ ③. $g \cdot v = \rho(g)(v)$

③ ⇒ ②. $\rho(g) = (m_g : v \mapsto gv)$

② ⇒ ① . choose a basis $(e_1 \dots e_n)$ of $V$, $\rho(g)(e_1 \cdots e_n) = (e_1 \cdots e_n) \cdot R(g)$

Two matrix rep'n $R, R'$ are conjugate, if $\exists P \in GL(n, F)$,

$$P R(g) P^{-1} = R'(g) \quad \forall g \in G.$$

Two $G$ rep'n $V, V'$ are isomorphic, if $f: V \to V'$

$$\begin{array}{ccc} G \times V & \to & V \\ \downarrow{\scriptstyle id \times f} & & \downarrow \\ G \times V' & \to & V' \end{array}$$

$$g \cdot f(v) = f(g \cdot v)$$

$F = \mathbb{C}$.

Defn: $R: G \to GL(n, \mathbb{C})$ is called (unitary representation) if $\operatorname{Im} R \subset U(n)$.

$V_{\mathbb{C}}$, $\langle\, ,\, \rangle : V \times V \to \mathbb{C}$ is called Hermitian form.

Hermitian form

① $\langle v, w_1 + w_2 \rangle = \langle v, w_1 \rangle + \langle v, w_2 \rangle$

② $\langle v, \lambda w \rangle = \lambda \langle v, w \rangle$

③ $\langle v, w \rangle = \overline{\langle w, v \rangle}$

positive definite Hermitian form.

positive definite

④ $\langle v, v \rangle \geq 0$. if $\langle v, v \rangle = 0 \Rightarrow v = 0$ $\quad \langle v, v \rangle = |v|^2$

Fact: $W \subset V$. $W^{\perp} = \{ v \in V \mid \langle w, v \rangle = 0, \forall w \in W \}$

then $W \oplus W^{\perp} = V$.

Pf: $W \cap W^{\perp} = \{0\}$. ( from ④ )

(Gram-Schmit) . $v_1, v_2 \cdots v_m$ linearly independent

则 $v_1' = \frac{1}{|v_1|} v_1$ . $\tilde{v}_2 = v_2 - \langle v_1', v_2 \rangle v_1'$

$v_2' = \frac{1}{|\tilde{v}_2|} \tilde{v}_2$ , $\tilde{v}_i = v_i - \sum_{j=1}^{i-1} \langle v_j', v_i \rangle v_j'$

$v_i' = \frac{1}{|\tilde{v}_j|} \tilde{v}_j$

Then $v_1' \cdots v_m'$ , $\langle v_i', v_j' \rangle = \delta_{ij}$

Choose $v_1', \cdots v_m'$ orthonormal basis of $W$ , then

$$\forall \ v = \underbrace{\sum \langle v_i', v \rangle v_i'}_{w} + (v - w)$$

则 $\langle v_i', v \rangle = \langle v_i', w \rangle$ , $\Rightarrow \langle v_i', v - w \rangle = 0$

$v - w \in W^\perp$ .

Cor : $(W^\perp)^\perp = W$ .

$$U(V) = \{ A \in GL(V) \mid \langle Av, Aw \rangle = \langle v, w \rangle \}$$

Defn : $\rho : G \to GL(V)$ unitary , if $\mathrm{Im}\,\rho \subset U(V)$

Thm: $G$ finite , $G \overset{\rho}{\curvearrowright} V$ repn , $\exists$ positive definite Hermitian form on $V$, s.t. $\rho$ is unitary.

Pf : Averaging under $G$-operation.

choose $\langle \cdot , \cdot \rangle : V \times V \to \mathbb{C}$ Hermitian form. positive definite

Define. $\langle v, w \rangle_G = \frac{1}{|G|} \sum_{g \in G} \langle gv, gw \rangle$

Then ① $\langle , \rangle_G$ 是一个 Hermitian form

② $\forall h \in G$, $\langle hv, hw \rangle_G = \frac{1}{|G|} \sum_{g \in G} \langle ghv, ghw \rangle$

$m_h : G \to G$ bijection
$g \mapsto g_h$
$= \frac{1}{|G|} \sum_{g \in G} \langle gv, gw \rangle$

$= \langle v, w \rangle_G$

V finite dimensional

Defn: $W \subset V$. is called $G$-invariant, $\Leftrightarrow$

$\forall g \in G$, $g(W) \subset W$ ($\Leftrightarrow g(W) = W$)

Why! finite dim'l case, may not hold.

Thm: G repn. V, $\dim V < \infty$, , $W \subset G$ invariant.

Then, $\exists W'$, G-invariant, $W \oplus W' = V$

Pf: choose $\langle , \rangle$ G-inv, Hermitian form

Claim: $W' = W^\perp$ G-invariant.

$\forall w \in W$, $v \in W^\perp$, $\langle w, gv \rangle = \langle g g^{-1} w, gv \rangle$

$= \langle g^{-1} w, v \rangle = 0$

$\Rightarrow gv \in W^\perp$

$F \neq \mathbb{C}$ counter ex. $G = (\mathbb{F}_p, +)$, $F = \mathbb{F}_p$. $n = 2$, $V = \mathbb{F}_p^2$

$R : G \to GL(2, \mathbb{F}_p)$

$x \mapsto \begin{bmatrix} 1 & x \\ & 1 \end{bmatrix}$

$W = \left\{ \begin{pmatrix} y \\ 0 \end{pmatrix} \middle| y \in \mathbb{F}_p \right\}$

$G$-invariant.

No $W'$ $G$-inv, s.t. $W \oplus W' = V$.

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Irreducible rep'n

Defn: $V \neq \{0\}$ has no $G$-invariant subspace other than $\{0\}$, $V$.