Ex of group rep'n :

Trivial: $G \to GL(1,\mathbb{C}) = \mathbb{C}^\times$

$g \mapsto 1.$

$D_n \to O(2) \subset GL(2,\mathbb{R}) \subset GL(2,\mathbb{C})$

$$D_n = \langle a, b \mid a^n = e,\ b^2 = e,\ bab^{-1} = a^{-1} \rangle \to GL(2,\mathbb{C})$$

$$a \longmapsto \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$

$$b \longmapsto \begin{pmatrix} 1 & \\ & -1 \end{pmatrix}$$

$\boxed{\text{regular repn}}$

$$\mathbb{C}[G] = \{ \sum_{g \in G} a_g g \mid a_g \in \mathbb{C} \}$$

$G \curvearrowright \mathbb{C}[G]$ $\qquad$ <u>$\mathbb{C}$-linear</u>

$$h \cdot (\sum_{g \in G} a_g \cdot g) = \sum_{g \in G} a_g (h \cdot g)$$

|

$\downarrow$

$G \curvearrowright G$ ,

$G \to S_G \to GL(\mathbb{C}[G])$

or $G \curvearrowright X$. $\# X = n$

extends to $F^X = \{ \sum_{x \in X} a_x x \mid a_x \in \mathbb{C} \}$.

$\Rightarrow G \to GL(n, F)$

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$S_n \curvearrowright \{e_1, \ldots, e_n\}$.

$S_n \curvearrowright \mathbb{C}^n$ , $\langle x, y \rangle = \bar{x}^T y$

$W = \mathrm{span}(e_1 + \cdots + e_n)$ $S_n$ invariant subspace.

Invariant Hermitian form $\langle , \rangle$

$\Rightarrow W^\perp = \{ \sum a_i e_i \mid \sum a_i = 0 \}$

Later we will see $W^{\perp}$ irreducible

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Direct sum of rep'ns

Quotient rep'ns, kernel, Image.

Dual rep'ns.

$\boxed{G \curvearrowright \mathrm{Hom}_{\mathbb{C}}(V, W)}$

Defn: $V \oplus W$, $g(v, w) = (gv, gw)$

In terms of matrix form

$$\begin{bmatrix} R_V(g) & 0 \\ 0 & R_W(g) \end{bmatrix}$$

(Semisimplicity)

G-rep'n $V$ is isomorphic to a direct sum of irreducible rep'n.

Pf: If $V$ has G-invariant subspace $W$ and $W \neq 0$, $W \neq V$ then choose $W^\perp$, under a G-invariant Hermitian form, $\Rightarrow$ $V = W \oplus W^\perp$. Induction on $\dim V$.

Defn : $W$ $G$-invariant subspace.

$V/W$ has $g$ operation by

$$g\cdot(v+W) = g.v + W$$

In terms of matrix

$$R_V(g) = \begin{bmatrix} R_W(g) & * \\ 0 & R_{(V/W)}(g) \end{bmatrix}$$

Defn: ($G$-homomorphism)

$f: V \to W$ $\quad f(g.v) = g\,f(v)$

$f$ $G$-linear

Then $\ker f$ $G$-invariant subspace $\subset V$

$\operatorname{Im} f \subset W$.

$$\operatorname{Im} \cong V/\ker f \quad \text{as } G\text{-rep'n}$$

Application: Schur's lemma (The most important lemma)

Let: $\mathrm{Hom}_G(V, W) = \{T : V \to W \mid T(gv) = gT(v)\}$
$\downarrow$
G-vector space

(Schur) If $V$, $W$ are irreducible,

Then $\dim \mathrm{Hom}_G(V, W) = \begin{cases} 1, & V \cong W \\ 0, & V \not\cong W \end{cases}$

Pf: $T \neq 0 \in \mathrm{Hom}_G(V, W)$, then

$\mathrm{Im}\, T \neq 0$, $\Rightarrow$ $W = \mathrm{Im}\, T$,

$\ker T \neq V$, $\Rightarrow$ $\ker T = 0$.

$V = W$ $T \in \mathrm{Hom}_G(V, V)$. Choose $V_\lambda$

$\lambda$-eigenspace of $T$, $\Rightarrow$ $V_\lambda = V$. by

$(T - \lambda\, \mathrm{Id}) \in \mathrm{Hom}_G(V, V)$ $\square$

Defn : $V^* = \mathrm{Hom}_{\mathbb{C}}(V, \mathbb{C}) = \{ f : V \to \mathbb{C} \mid f \ \mathbb{C}\text{-linear} \}$

$$(g \cdot f)(v) = f(g^{-1} v)$$

In terms of matrix. choose

basis $B$ : $v_1 \cdots v_n$ and

dual basis $B^*$ : $f_1 \cdots f_n$

$$f_i(v_j) = \delta_{ij} = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases}$$

Then $R_{V^*}(g) = \left( (R_V(g))^T \right)^{-1}$

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More generally. $V, W$ $G$-repn

$T \in \mathrm{Hom}_{\mathbb{C}}(V, W)$

$g \cdot T(v) = g\, T(g^{-1} \cdot v)$

check this defines a $G$-repn.

$\boxed{\text{In terms of matrix??}}$

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Tensor product.

$V \otimes W$, $V$ has basis $v_1 \cdots v_n$

$W$ has basis $w_1 \cdots w_m$

$\underline{v_i \otimes w_j}$ is a basis of $V \otimes W$

any two vectors $v \in V$, $w \in W$.

write $v = \sum a_i v_i$, $w = b_j w_j$.

$v \otimes w = \sum a_i b_j \, v_i \otimes w_j$

any two other basis $v_1' \dots v_s'$.

$w_1' \dots w_m'$.

$v_i' \otimes w_j'$ can be written as linear combinations of $v_i \otimes w_j$.

use the same rule, $v_i \otimes w_j$ can be written as linear combinations of $v_i' \otimes w_j'$.

So elements in $V \otimes W$ has the form $\sum a_{ij} v_i \otimes w_j$ or $\sum a_{ij}' v_i' \otimes w_j'$

and they are related by linear combinations

( $V \otimes W$ does not depend on the choices of basis )

Lemma: $T: V^* \otimes W \to \mathrm{Hom}_{\mathbb{C}}(V, W)$

$$f \otimes w \longmapsto (v \mapsto f(v)\cdot w)$$

or $$\sum a_{ij} f_j \otimes w_i \longmapsto (v \mapsto \sum a_{ij} f_j(v)\cdot w_i)$$

is a linear isomorphism.

Pf: Surjective: $\mathcal{B}: (v_1, \dots, v_n)$ basis of $V$

$\mathcal{C}: (w_1, \dots, w_m)$ basis of $W$

$$F(v_1 \dots v_n) = (w_1, \dots, w_m)\cdot (a_{ij})_{m\times n}$$

Then choose $f_1 \dots f_n$ dual basis of $\mathcal{B}$,

$$T\left(\sum_{i,j} a_{ij} f_j \otimes w_i\right)(v_k)$$

$$= \sum_i a_{ik}\cdot w_i.$$

$\square$