$\underline{V \otimes W}$ $\qquad g \cdot (v \otimes w) = (g \cdot v) \otimes (g \cdot w)$

$v_1 \dots v_n$ basis of V.

$w_1 \dots w_m$ basis of W.

$$g\, v_j = \sum_i a_{ij} v_i \qquad A = (a_{ij})$$

$$g\, w_k = \sum_l b_{lk} w_l \qquad B = (b_{lk})$$

$$g(v_j \otimes w_k) = \left(\sum a_{ij} v_i\right) \otimes \left(\sum b_{lk} w_l\right)$$

$$= \sum_{i,l} a_{ij} b_{lk} (v_i \otimes w_l)$$

$$A \otimes B = \begin{pmatrix} a_{11} B & a_{12} B & \dots \\ & & \\ & & \end{pmatrix}$$

$$\mathrm{Tr}(A \otimes B) = \mathrm{Tr}(A) \cdot \mathrm{Tr}(B)$$

Character theory

$\rho : G \to GL(V)$ or $GL(n, \mathbb{C})$

$$\chi_\rho : G \to \mathbb{C}$$
$$g \mapsto Tr(\rho(g))$$

$\chi_\rho(e) = \dim_{\mathbb{C}} V$ since $\rho(e) = Id$

" χ_ρ determines ρ "

$\rho_1 \oplus \rho_2$, $\Rightarrow$ $\chi_{\rho_1 \oplus \rho_2} = \chi_{\rho_1} + \chi_{\rho_2}$

dual (complex conjugate) $\chi_{\rho^*} = \overline{\chi_\rho}$

$V \otimes W$, $\chi_{\rho_V \otimes \rho_W} = \chi_{\rho_V} \cdot \chi_{\rho_W}$

construct new characters by addition, multiplication

Ex: $V = \mathbb{C}[G]$ regular repn

$$\chi_{reg}(g) = \begin{cases} \# G & \text{if } g = e \\ 0 & \text{if } g \neq 0. \end{cases}$$

Def (class functions) $\mathbb{C}$- vector spa

$$\mathcal{C}(G) = \{ f : G \to \mathbb{C} \mid f(g) = f(hgh^{-1}) \ \forall h \in G,\ g \in G \}$$

Prop: $\dim_{\mathbb{C}} \mathcal{C}(G) = \#$ conjugacy classes in G

Prop: $\forall \rho : G \to GL(n, \mathbb{C})$

$$\chi_\rho \in \mathcal{C}(G).$$

Pf: $\mathrm{Tr}(PAP^{-1}) = \mathrm{Tr}(A)$

$\square$

Hermitian form on $\mathcal{C}(G)$

$$\langle f_1, f_2 \rangle = \frac{1}{\# G} \sum_{g \in G} \overline{f_1(g)} f_2(g)$$

$\{\rho_1, \rho_2 \dots \rho_r\} = \mathrm{Irr}_{\mathbb{C}}(G)$

$\chi_i = \chi_{\rho_i} \in \mathcal{C}(G)$

Thm: ① $\langle \chi_i, \chi_j \rangle = \delta_{ij}$ ($\Rightarrow$ linearly independent)
orthonormal

② $\chi_1 \dots \chi_r$ form a basis of $\mathcal{C}(G)$.

③ $n_i := \chi_i(e) \mid \# G$.

④ $\mathbb{C}[G] \cong \bigoplus_{i=1}^{r} \rho_i^{n_i}$

① Most important relation.

Cor1: If $\rho = \bigoplus_{i=1}^{r} \rho_i^{k_i} \Rightarrow k_i = \langle \chi_\rho, \chi_i \rangle$

only depends on ρ.

i.e. $\bigoplus \rho_i^{k_i} \cong \bigoplus \rho_i^{l_i} \Rightarrow l_i = k_i$.

Cor 2: ρ irreducible iff $\langle \chi_\rho, \chi_\rho \rangle = 1$.

Pf: if $\rho = \bigoplus_{i=1}^{r} \rho_i^{k_i}$. $\langle \chi_\rho, \chi_\rho \rangle = \sum_{i=1}^{r} k_i^2$ $\square$

③ also follows from ①.

$$\langle \chi_{reg}, \chi_i \rangle = \frac{1}{\# G} \sum \chi_{reg}(g) \cdot \chi_i(g)$$

$$= \chi_i(e) = \dim \rho_i$$

Pf of Thm:

Need the following interpretation of $\langle \chi_V, \chi_W \rangle$

Lemma:

$$\langle \chi_V, \chi_W \rangle = \frac{1}{\# G} \sum_{g \in G} \underline{\chi_{V^* \otimes W}(g)}$$

How to calculate $\sum \chi_\rho(g)$ for some rep'n ρ.

Lemma: $\rho: G \to GL(V)$, V^G or

$\mathrm{Inv}(G) = \{ v \mid g \cdot v = v, \text{ for all } g \in G \}$.

then $\frac{1}{\# G} \sum \chi_\rho(g) = \dim \mathrm{Inv}(G)$

Pf: Use an operator:

$$P = \frac{1}{\# G} \sum_{g \in G} \rho(g) \in \mathrm{Hom}_G(V, V)$$

① P commute with g operation (averaging)

$\forall h \in G$. $P \circ \rho(h) = \frac{1}{\# G} \sum_{g \in G} \rho(g)\rho(h) = P$

$\rho(h) \circ P = \frac{1}{\# G} \sum_{g \in G} \rho(h)\rho(g) = P$

$p \in \mathrm{Hom}_G(V, V)$

② $p^2 = p$. $\quad p \cdot p = \frac{1}{\#G} \sum_{h \in G} p \circ \rho(h) = p$

projection operator

③ $V = V_0 \oplus V_1$, $\quad V_0 = \ker p$

$V_1 = \operatorname{Im} p$ 1-eigenspace

$\Rightarrow \mathrm{Inv}(G) \subset V_1$. and

$\forall \; v \in V_1$, $\quad \underbrace{\rho(g) \circ p}_{p} \underbrace{(v)}_{v} = p(v) = v$

$\Rightarrow V_1 \subset \mathrm{Inv}(G)$

So $\quad \sum_{g \in G} \chi_\rho(g) = \mathrm{Tr}(p) = \dim_{\mathbb{C}} \mathrm{Inv}(G)$

Lemma: $\mathrm{Inv}_G(\mathrm{Hom}_{\mathbb{C}}(V, W)) = \mathrm{Hom}_G(V, W)$

pf: By defn $\square$

Pf of Thm: V, W irreducible, then.

$$\langle \chi_V, \chi_W \rangle = \frac{1}{\# G} \sum_{g \in G} \chi_{V^* \otimes W}(g)$$

$$= \dim_{\mathbb{C}} (\mathrm{Hom}_G(V, W))$$

$$\overset{\text{Schur}}{=} \begin{cases} 1 & V \cong W \\ 0 & V \not\cong W. \end{cases}$$

$\square$