Orthonormal $\Rightarrow$ linearly independent.

So $\#\{\chi_1 \ldots \chi_r\} \leq \dim_{\mathbb{C}} \mathcal{C}(G)$

$\Downarrow$

χ_i irreducible, distinct characters of G.

Next: $\chi_1 \ldots \chi_r$ generate $\mathcal{C}(G)$

$\forall\ \theta \in \mathcal{C}(G) \quad \theta: G \to \mathbb{C}$

$\chi_1 \ldots \chi_r$ orthogonal $\Rightarrow$

$$\tilde{\theta} = \theta - \sum_{i=1}^{r} \overline{\langle \theta, \chi_i \rangle} \chi_i \in \mathcal{C}(G)$$

and $\langle \tilde{\theta}, \chi_i \rangle = 0$.

Claim: $\tilde{\theta} = 0$.

Pf: For any $f: G \to \mathbb{C} \in \mathcal{C}(G)$

and $\rho: G \to GL(V)$

define $\rho_f = \frac{1}{\# G} \sum_{g \in G} \overline{f(g)} \cdot \rho(g)$

Then $\rho(h)\,\rho_f \circ \rho(h^{-1})$

$$= \frac{1}{\# G} \sum_{g \in G} \overline{f(g)}\, \rho(h)\rho(g)\cdot\rho(h^{-1})$$

$$= \frac{1}{\# G} \sum_{g \in G} \overline{f(hgh^{-1})}\, \rho(hgh^{-1})$$

$$= \rho_f .$$

$\Rightarrow \rho_f \in \mathrm{Hom}_G(V, V)$

If V irreducible $\Rightarrow \rho_f = \lambda \cdot \mathrm{Id}_V$

and $\lambda = \dfrac{\langle f, \chi_V \rangle}{\dim V}$ by $\mathrm{Tr}\,\rho_f = \langle f, \chi_V \rangle$

$$\rho_{reg} = \bigoplus_{i=1}^{r} \rho_i^{\oplus n_i}$$

Calculate $(\rho_{reg})_{\tilde{\theta}}$.

$\langle \tilde{\theta}, \chi_i \rangle = 0$, $\Rightarrow (\rho_{reg})_{\tilde{\theta}} = 0$.

On the other hand.

$$\rho_{reg}(e) = \frac{1}{\# G} \sum_{g \in G} \overline{\vec{\theta}(g)} \cdot (g \cdot e)$$

$$= \frac{1}{\# G} \sum_{g \in G} \overline{\vec{\theta}(g)} \cdot g = 0$$

$$\Rightarrow \vec{\theta}(g) = 0$$

$\square$

$\chi_i(e) \mid \# G$ requires more on algebraic number theory.

Calculate the character table for G

$G = C_n = \mathbb{Z}/n\mathbb{Z}$.

Then $\chi_l : G \to \mathbb{C}^{\times}$,

$$\bar{k} \longmapsto e^{\frac{2\pi\sqrt{-1}}{n} \cdot kl} \qquad e^{\frac{2\pi\sqrt{-1}}{n}} = \xi_n$$

$\chi_0, \chi_1 \cdots \chi_{n-1}$

List all the elements

$e, \quad a, \quad \ldots \quad a^{n-1}$

$\chi_0 \quad 1, \quad 1 \quad \ldots \quad 1$

$\chi_1 \quad 1, \quad \zeta_n \quad \zeta_n^2 \quad \ldots \quad \zeta_n^{n-1}$

$\chi_2 \quad 1, \quad \zeta_n^2 \quad \zeta_n^4 \quad \zeta_n^6 \quad \ldots \quad \zeta_n^{2(n-1)}$

$\vdots$

$\chi_{l-1} = \chi_1^{l-1}$

S_3. conjugacy classes

	(1),	(12)	(123)
χ_1	1	1	1
χ_2	1	-1	1
χ_3	2	0	-1.
χ_V	3,	1,	0.

$\langle \chi_V, \chi_V \rangle = \frac{1}{6}(9+3) = 2 \rightarrow \boxed{\text{subtract } \chi_1}$

D_n conjugacy classes

	e,	$\{x, x^{-1}\}$,	$\{x^2, x^{-2}\}$ ---	$\{xy, x^2y \dots x^{n-1}y\}$
χ_1	1	1	1	
χ_2	1	1	1 . -	1
χ_3				-1
χ_4				
$\vdots$				

$\boxed{D_n \xrightarrow{\det} \{\pm 1\}}$

$x \longmapsto 1$

$y \longmapsto -1.$

$D_n \to O(2) \to GL(2, \mathbb{C})$

$x \longmapsto \begin{pmatrix} \cos\frac{2\pi}{n} & -\sin \\ \sin & \cos \end{pmatrix}$

$y \longmapsto \begin{pmatrix} 1 & \\ & -1 \end{pmatrix}$

$\chi(x) = 2\cos\frac{2\pi}{n} < 2$

$\chi(y) = 0$

irreducible for $n \geq 3$

$\langle \chi, \chi \rangle < \frac{1}{2n}(4 + 2n)$

$\underline{S_4}$

	(1)	(12)	$(12)(34)$	(123)	(1234)
χ_1	1	1	1	1	1
χ_2	1	-1	1	1	-1
χ_3	3	1	-1	0	-1
$\overset{\pi}{\chi}$	4	2	0	1	0
$\chi_4 = \chi_2 \cdot \chi_3$	3	-1	-1	0	1
$\chi =!$	2	0	a	b	0.

Only one left

$$1^2 + 1^2 + 3^2 + 3^2 + \chi(e)^2 = \# G = 24$$

$$\Rightarrow \chi(e) = 2.$$

$\chi_5 \cdot \chi_2 = \chi_5 \Rightarrow \chi_5((12)) = 0.\quad \chi_5((12\,34)) = 0$

$\langle \chi_4 \cdot \chi_5 \rangle = \frac{1}{24} \cdot (6 - 3a) = 0$

$\Rightarrow a = 2$

$\langle \chi_5, \chi \rangle = 0 \Rightarrow \frac{1}{24} (8 + 8b) = 0$

$\Rightarrow b = -1$

Second orthogonal relation:

$$\begin{array}{c|cccc} & C_1 & C_2 & \cdots & C_r \\ \hline \chi_1 & & & & \\ \vdots & & & & \\ \chi_r & & & & \end{array}$$

$\chi_i(C_j) = a_{ij}$

$$\frac{1}{\# G} \bar{A} \cdot \begin{pmatrix} \# C_1 & & \\ & \ddots & \\ & & \# C_r \end{pmatrix} A^T = I$$

$$\Rightarrow \frac{1}{\# G} A^T \bar{A} \cdot \begin{bmatrix} \# C_1 & & \\ & \ddots & \\ & & \# C_r \end{bmatrix} = I$$

$$\Rightarrow \sum_{k=1}^{r} \chi_k(C_i) \cdot \overline{\chi_k(C_j)} \cdot \# C_j = \# G \, \delta_{ij}$$

$$\Rightarrow \boxed{\sum_{k=1}^{r} \chi_k(C_i) \overline{\chi_k(C_j)}} = \underline{\frac{\# G}{\# C_j}} \, \delta_{ij}$$

centralizer of $g_j \in C_j$

One column $\begin{matrix} \chi_1(C) \\ \vdots \\ \chi_r(C) \end{matrix}$

Symmetric rep'n and Alternating rep'n.

$V \otimes V \circlearrowleft S_2$

$\parallel$

$S^2(V) = \mathrm{Span}(v \otimes w + w \otimes v)$

$\Lambda^2(V) = \mathrm{Span}(v \otimes w - w \otimes v)$

$v_1 \cdots v_n$. basis of V.

$S^2(V)$ has basis $v_i \otimes v_j + v_j \otimes v_i = v_i v_j$

$1 \le i \le j \le n$.

$\Lambda^2(V)$ — ——

$v_i \otimes v_j - v_j \otimes v_i = v_i \wedge v_j$

$1 \le i < j \le n$.

$G \curvearrowright V \Rightarrow G \curvearrowright V \otimes V$. and

commute with $S_2 \circlearrowleft$ so

$V \otimes V$ is a $G \times S_2$ rep'n

$\parallel$

$\Downarrow$

$\Lambda^2 V$ are invariant under G.
$S^2 V$

Prop: $\chi_{S^2(V)}(g) = \frac{1}{2}(\chi(g)^2 + \chi(g^2))$

$\chi_{\Lambda^2(V)}(g) = \frac{1}{2}(\chi(g)^2 - \chi(g^2))$

Pf: $\forall g$, $\exists$ basis $v_1, \dots, v_n$

s.t. $g v_i = \lambda_i v_i$

$\Rightarrow g(v_i \otimes v_j) = \lambda_i \lambda_j (v_i \otimes v_j)$

$\Rightarrow \dots$ $\square$

$\boxed{\text{Apply to } S_4}$ or S_5