SU(2) and orthogonal repn

$$U(2) = \left\{ A = \begin{pmatrix} a, b \\ c, d \end{pmatrix} \in M_2(\mathbb{C}) \;\middle|\; \bar{A}^T \cdot A = I \right\}$$

$$SU(2) = \left\{ A = \begin{pmatrix} a \cdot b \\ c, d \end{pmatrix} \in M_2(\mathbb{C}) \;\middle|\; \begin{array}{l} \bar{A}^T A = I \\ \det A = 1 \end{array} \right\}$$

$$A^{-1} = \begin{pmatrix} d & -b \\ -c & a \end{pmatrix} = \bar{A}^T$$

$$\Rightarrow \quad a = \bar{d}, \quad b + \bar{c} = 0$$

$$A = \begin{bmatrix} a & b \\ -\bar{b} & \bar{a} \end{bmatrix}, \quad \underline{\bar{a}a + b\bar{b} = 1}$$

$a = x + y\sqrt{-1}$, $b = z + w\sqrt{-1}$.

$|a|^2 + |b|^2 = x^2 + y^2 + z^2 + w^2 = 1$. S^3

Goal: $SU(2) \xrightarrow{2:1} SO(3)$

$\downarrow$

Real rep'n. and group homo.

$A \in SU(2)$, consider

$W = \{B \in M_2(\mathbb{C}),\ s.t.\ \mathrm{tr}B = 0.\ \underbrace{B^T + \bar{B} = 0}\}$

$\downarrow$

Then $ABA^{-1} \in W$. $\underline{\dim_{\mathbb{R}} = 3}$

$Tr(ABA^{-1}) = Tr(B)$

$(ABA^{-1})^T = (AB\bar{A}^T)^T = \bar{A} B^T A^T$

$= -\bar{A}\bar{B}\bar{A}^{-1}$

$SU(2) \curvearrowright W$ and $\mathbb{R}$-linear

$\Rightarrow SU(2) \longrightarrow GL(3, \mathbb{R})$

W has an inner product.

$\langle B, B' \rangle = -\frac{1}{2} Tr(BB')$

W has basis

$$B_1 = \begin{bmatrix} \sqrt{-1} & \\ & -\sqrt{-1} \end{bmatrix}, \quad B_2 = \begin{bmatrix} & 1 \\ -1 & \end{bmatrix}, \quad B_3 = \begin{bmatrix} & \sqrt{-1} \\ \sqrt{-1} & \end{bmatrix}$$

$B_1^2 = -I$, $B_2^2 = -I$, $B_3^2 = -I$

$$B_1 B_2 = \begin{pmatrix} & \sqrt{-1} \\ \sqrt{-1} & \end{pmatrix} = B_3$$

$B_2 B_3 = B_1$, $B_3 B_1 = B_2$

$\langle \, , \, \rangle$ positive definite

B_1, B_2, B_3 orthonormal basis

$\rho : SU(2) \longrightarrow O(3)$

(connected so $\operatorname{Im} \rho \subset SO(3)$)

Prop: $\ker\rho = \{\pm I\}$ and $\operatorname{Im}\rho = SO(3)$

Pf: $\pm I \subset \ker\rho$

$$A B_i A^{-1} = B_i \Rightarrow A B_i = B_i A$$

$i = 1, 2, 3$, $B_1 \Rightarrow A \begin{pmatrix} 1 & \\ & -1 \end{pmatrix} = \begin{pmatrix} 1 & \\ & -1 \end{pmatrix} A$

$$\begin{pmatrix} a & b \\ -\bar{b} & \bar{a} \end{pmatrix} \begin{pmatrix} 1 & \\ & -1 \end{pmatrix} = \begin{pmatrix} 1 & \\ & -1 \end{pmatrix} \begin{pmatrix} a & b \\ -\bar{b} & \bar{a} \end{pmatrix}$$

$$\begin{pmatrix} a & -b \\ -\bar{b} & -\bar{a} \end{pmatrix} = \begin{pmatrix} a & b \\ \bar{b} & -\bar{a} \end{pmatrix} \Rightarrow b = 0$$

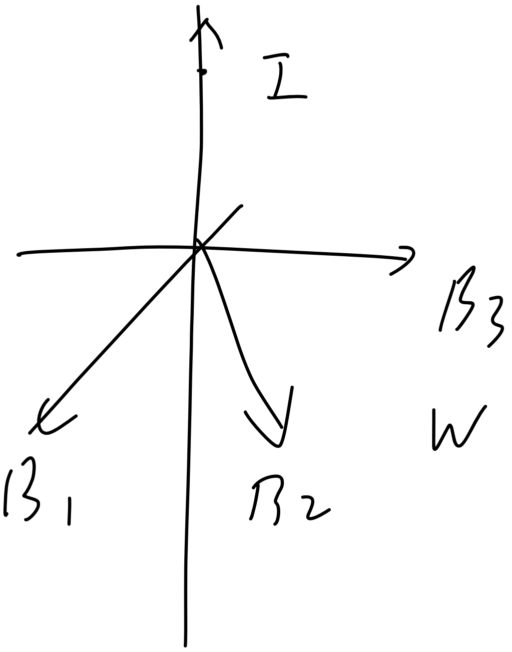
$B_2 \Rightarrow$

$$\begin{pmatrix} a & 0 \\ 0 & \bar{a} \end{pmatrix} \begin{pmatrix} & 1 \\ -1 & \end{pmatrix} = \begin{pmatrix} & 1 \\ -1 & \end{pmatrix} \cdot \begin{pmatrix} a & \\ & \bar{a} \end{pmatrix}$$

$$\begin{pmatrix} & a \\ -\bar{a} & \end{pmatrix} = \begin{pmatrix} & \bar{a} \\ -a & \end{pmatrix} \Rightarrow a = \bar{a}$$

$|a|^2 = 1 \Rightarrow a = \pm 1$.

<u>$\operatorname{Im} \rho$:</u> <u>rotation</u>



<u>Claim:</u>

$SU(2) \curvearrowright W$

action transitive

on $S^2 \subset W$

$$S^2 = \{ a_1 B_1 + a_2 B_2 + a_3 B_3 \mid \textstyle\sum a_i^2 = 1 \}$$

$$= C = \{ B \in SU(2) \mid \operatorname{Tr} B = 0 \}$$

eigenvalues of $A \in SU(2)$ are

$\lambda, \bar{\lambda}$, $\lambda \cdot \bar{\lambda} = 1$.

$\operatorname{Tr} A = \lambda + \bar{\lambda}$.

If $\lambda, \bar{\lambda}$ are distinct.

then eigenvectors v_1, v_2. are orthogonal. rescale to orthonormal

$$\Rightarrow \quad A\underbrace{(v_1, v_2)}_{P} = (v_1, v_2)\begin{pmatrix} \lambda & \\ & \bar{\lambda} \end{pmatrix}$$

rescale P to $\det P = 1$

$$\Rightarrow \quad P \in SU(2). \quad PAP^{-1} = \begin{pmatrix} \lambda & \\ & \bar{\lambda} \end{pmatrix}$$

Apply to $B \in SU(2)$, $\operatorname{tr} B = 0$

$$\Rightarrow \quad PBP^{-1} = \begin{pmatrix} \sqrt{-1} & \\ & -\sqrt{-1} \end{pmatrix} = B_1$$

Stab_{B_1} in $SU(2)$ is $\begin{pmatrix} a & 0 \\ 0 & \bar{a} \end{pmatrix}$

and $a = e^{\sqrt{-1}\theta}$, then A_θ

$$A_\theta = \cos\theta \cdot \begin{pmatrix} 1 & \\ & 1 \end{pmatrix} + \sin\theta \begin{pmatrix} \sqrt{-1} & \\ & -\sqrt{-1} \end{pmatrix}$$
$$= \cos\theta\, I + \sin\theta\, B_1$$

$$A_\theta^{-1} = \cos\theta \, I - \sin\theta \, B_1$$

$$A_\theta B_2 A_\theta^{-1} = (\cos\theta \, I + \sin\theta \, B_1) \cdot B_2 (\cos\theta \, I - \sin\theta \, B_1)$$

$$= (\cos^2\theta - \sin^2\theta) B_2 + 2\sin\theta\cos\theta \, B_3$$

$$A_\theta B_3 A_\theta^{-1} = (-2\sin\theta\cos\theta) B_2 + (\cos^2\theta - \sin^2\theta) B_3$$

$$\rho(A_\theta) \cdot (B_1, B_2) = (B_1, B_2) \cdot \underbrace{\begin{pmatrix} \cos 2\theta & -\sin 2\theta \\ \sin 2\theta & \cos 2\theta \end{pmatrix}}_{\text{rotation by } 2\theta}$$

Then $P A_\theta P^{-1}$ can realite all rotations by any angle.

$\operatorname{Im} \rho = SO(3)$

D.

Now any finite subgroup G of $SU(2)$ has the following description

$$\rho: SU(2) \longrightarrow SO(3)$$

$$\rho|_G : G \longrightarrow SO(3)$$

$$\rho: G \xrightarrow{2:1} \mathrm{Im}\, G \quad \text{or} \quad \rho: G \underset{1,0}{\xrightarrow{\cong}} \mathrm{Im}\, G$$

<u>Classification</u> and <u>rep'ns.</u>