McKay graph.

connected.

Thm: If $\rho: G \to GL(V)$ is faithful, the every irreducible rep'n appears in the irreducible decomposition of $V^{\otimes n}$ for some n.

Cor: If V faithful, then the McKay graph is connected.

$(\mathbb{1} \otimes V^{\otimes n}) \to$ vertex $\rho_1 = \mathbb{1}$ is connected to any irreducible rep'n.

Pf of Thm:

Use $(W_1 \oplus W_2) \otimes V$

$$= W_1 \otimes V \oplus W_2 \otimes V$$

Set $W = V \oplus \mathbb{1}$.

$$W^{\otimes N} = \bigoplus_{\text{some } n} V^{\otimes n}$$

$$\langle \chi_i , \chi_W \rangle = \frac{1}{\# G} \sum_{g \in G} \overline{\chi_i(g)}\, \chi_W(g)$$

$|\chi_W(g)| \leq \dim W$ and when equality holds $g = e$.

So $\langle \chi_i, \chi_W \rangle \neq 0$ for N large.

$\Rightarrow$ χ_i appears in irreducible decomp of χ_W $\square$

ADE classification.

Γ graph $\quad i \xrightarrow{n_{ij}} j$, $\quad r = \#$ of vertices

undirected $n_{ij} = n_{ji}$, no loop $n_{ii} = 0$

Matrix $M = (n_{ij})$,

$M^T = M$, define bilinear form by $(2I - M)$

$B : \mathbb{Z}^r \times \mathbb{Z}^r \to \mathbb{Z}$

$$\begin{pmatrix} x_1 \\ \vdots \\ x_r \end{pmatrix} \begin{pmatrix} y_1 \\ \vdots \\ y_r \end{pmatrix} \longmapsto 2\sum_{i=1}^{n} x_i y_i - \sum_{i,j}\sum n_{ij} x_i y_j$$

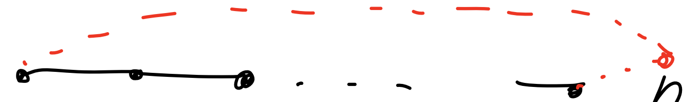
or quadratic form

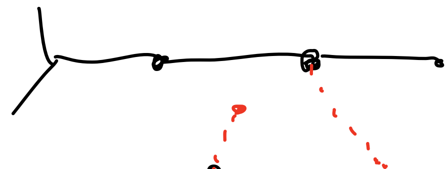
$$q_\Gamma(x) = \frac{1}{2} B(x,x) = \sum x_i^2 - \sum_{i<j} n_{ij} x_i y_j$$

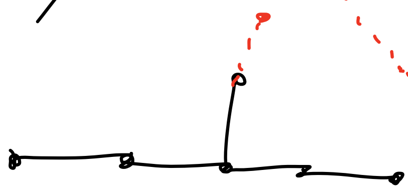
simply laced, $n_{ij} \in \{0, 1\}$

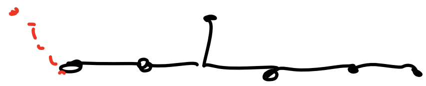
Γ connected, simply laced, no loop,

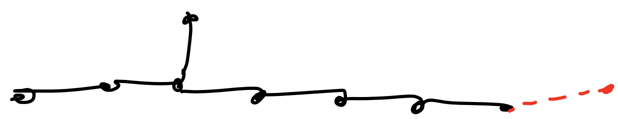
q positive definite. then Γ is from ADE.

A_n  $n \geq 1$

D_n  $n \geq 3$

E_6 

E_7 

E_8 

positive semi definite, then

$\tilde{A}_n$ $\tilde{E}_7$, $\tilde{E}_8$ (adding one node)

$\tilde{D}_n$

$\tilde{E}_6$

red node

Lemma 1: If M has eigenvector (Γ connected)

$$v = \begin{pmatrix} v_1 \\ \vdots \\ v_r \end{pmatrix} \quad \text{s.t.} \quad Mv = 2v,$$

($v \in \ker(2I - M)$), and $v_i \geq 0$,

then $v_i > 0$, and q semi-positive defin

and $q(w) = 0 \iff w \in \mathbb{R}v$.

Pf: $\sum_j n_{ij} v_j = 2 v_i$

$v_i = 0$ implies $v_j = 0$ for

$$\underset{i}{\circ} \longrightarrow \underset{j}{\circ}, \quad \text{so } v_i = 0 \Rightarrow$$

$v_j = 0$ for all j adjacent to i.

So $\underline{v_i > 0}$

$$\sum_i \sum_j n_{ij} v_i v_j \left(\frac{x_i}{v_i} - \frac{x_j}{v_j} \right)^2$$

$$= \sum_i \sum_j n_{ij} \left(\frac{v_j}{v_i} x_i^2 + \frac{v_i}{v_j} x_j^2 \right)$$

$$- 2 \sum_{i,j} n_{ij} x_i x_j$$

$$= 4 \left(\sum_i x_i^2 - \sum_{i<j} n_{ij} x_i x_j \right) \geq 0$$

$$\left(\sum_j n_{ij} v_j = 2 v_i \right)$$

equality holds on $\frac{x_i}{v_i} = \frac{x_j}{v_j} = \lambda$.

$x_i = \lambda v_i$.

Lemma 2: Γ' subgraph of Γ, then

$q_\Gamma \geq 0 \Rightarrow q_{\Gamma'} \geq 0$.

Pf: Γ_0 vertex set of Γ.
Γ_0' vertex set of Γ'.

$n_{ij}' \leq n_{ij}$

Assume $\Gamma_0 = \Gamma_0'$

$$q_\Gamma = \sum_{i \in \Gamma_0} x_i^2 - \sum_{i,j} n_{ij} x_i x_j$$

$$q_{\Gamma'} = \left(\sum_{i \in \Gamma_0'} x_i^2 - \sum_{i,j} n_{ij}' x_i x_j \right)$$

If $q_{\Gamma'} \not\geq 0$, $q_{\Gamma'}(y) < 0$

$y = (y_1 \cdots y_k\ 0 - - 0)$

$q_\Gamma(|y_1| \cdots |y_k|\ 0 - - 0) \leq q_{\Gamma'} < 0$

Check $ADE > 0$. $\widetilde{ADE} \geq 0$

by Lemma 1 and the only graph contains no $\widetilde{ADE}$ are ADE, so this implies $\widetilde{ADE}$ are the only ones that are semi-positive definite.