Sylow p-subgroups

$\# G = p^n \cdot m$. p prime number

$p \nmid m$.

H is called Sylow p-subgroup, if $\# H = p^n$.

Thm (Sylow)

① Existence, $\exists$ Sylow p-subgroup

② Unique up to conjugation.

For H_1, H_2 both Sylow p-subgroups of G, $\exists g \in G$, s.t. $g H_1 g^{-1} = H_2$

③ $a_p = \#$ Sylow p-subgroups

$a_p \equiv 1 \pmod p$. $a_p \mid m$.

Ex: $G = GL(n, \mathbb{F}_p)$, $\mathbb{F}_p = \mathbb{Z}/p\mathbb{Z}$, p prime

$$U = \left\{ \begin{bmatrix} 1 & & * \\ & \ddots & \\ & & 1 \end{bmatrix} \right\} \subset G.$$

$$\# G = (p^n - 1)(p^n - p) \cdots (p^n - p^{n-1})$$

$$\# U = (p^n - 1)(p^{n-1} - 1) \cdots (p-1) \cdot p^{\frac{n(n-1)}{2}}$$

$$\# \quad = p^{\frac{n(n-1)}{2}}$$

$U \subset G$ Sylow p-subgroup.

Linear rep'n of S_n

$S_n \longrightarrow GL(n, \mathbb{F}_p)$ group h-mo

$\sigma \longmapsto (e_{\sigma(1)} \cdots e_{\sigma(n)})$

permute the column vectors.

Then $S_n \cong$ subgroup of $GL(n, \mathbb{F}_p)$

Any group $G \cong$ subgroup of
finite

$GL(n, \mathbb{F}_p)$
for some n.

Existence:

Lemma: $H \subset G$ subgroup,

U sylow p-subgroup of G, then $\exists g \in G$,

s.t. $\underbrace{gUg^{-1}}_{\text{appears as stab}} \cap H$ sylow p-subgroup of H.

Pf: Consider $G \curvearrowright G/U$

restrict to $H \curvearrowright G/U$,

Then $\# G/U \not\equiv 0 \pmod{p}$

$$G/U = O_1 \sqcup O_2 \sqcup \cdots \sqcup O_l.$$

$\exists\ O_i$ s.t. $\# O_i \not\equiv 0 \pmod p$

$O_i = g_i U$, then

$\mathrm{Stab}_{g_i U}$ under H-operation is $g_i U g_i^{-1} \cap H$

$$\Rightarrow \# O_i = \frac{\# H}{\# g_i U g_i^{-1} \cap H} \not\equiv 0 \pmod p$$

$\Rightarrow$ $g_i U g_i^{-1} \cap H$ sylow p-subgroup of H

Cor: ① Existence by

$$G \hookrightarrow GL(n, \mathbb{F}_p)$$

② Conjugation, apply to $H_1 \subset G$, $U = H_2 \subset G$.

3. counting

$X = \{H_1, \dots, H_l\}$ all Sylow p-subgroups

$G \circlearrowright X$ transitive by conjugation.

So $l = \dfrac{\# G}{\# \mathrm{Stab}_{H_1}}$ $\quad \mathrm{Stab}_{H_1} \supset H_1$

$\Rightarrow l \mid m$.

Restrict to H_1, then

$X = O_1 \sqcup O_2 \cdots \sqcup O_k$

$O_1 = \{H_1\}$. If $O_i = \{H_i\}$,

Look at $N_G(H_i)$

$= \{g \in G \mid g H_i g^{-1} = H_i\}$.

Then H_1, H_i are both Sylow p-subgroups in $N_G(H_i)$,

and H_i normal in $N_G(H_i)$

$\Rightarrow H_1 = H_i$

$\Rightarrow \# O_1 = 1$, $\# O_i = p^{l_i}$, $(l_i \geq 1)$, $i \geq 2$

So $l \equiv 1 \pmod{p}$

Ex: $\# G = 35$.

$\Rightarrow a_5 = 1$, $a_7 = 1$ so

H Sylow 5-subgroup. K Sylow 7-subgroup

H, K normal.

$H \cap K = \{e\}$ because g.c.d$(5,7) = 1$.

$$\Rightarrow \quad H \times K \xrightarrow[\cong]{\text{Hw2}} HK \quad \text{and} \quad G = HK$$

$$G \cong \mathbb{Z}/5\mathbb{Z} \times \mathbb{Z}/7\mathbb{Z}, \left(\cong \mathbb{Z}/35\mathbb{Z} \right)$$

Chinese remainder Thm