Generators and relations

D_n $\quad \langle a, b \rangle \quad \left\{ \begin{array}{l} bab^{-1} = a^{-1} \\ a^n = e. \\ b^2 = e. \end{array} \right.$

First every element in D_n can be written as product of $a. b$.

and use $ba = a^{-1}b = a^{n-1}b$

any product $= a^{i_1} b^{j_1} a^{i_2} b^{j_2} \cdots \in D_n$ has the form $g = a^i b^j$.

This determines the structure of D_n

More generally, we have the defn of free groups

Defn (word) letter $x_1 \cdots x_n$, word $\underline{(x_{i_1})^{k_1} \cdots (x_{i_l})^{k_l}}$ $k_1, \dots, k_l \in \mathbb{Z}$.

$\underline{i_j \neq i_{j+1}}$ (reduced)

Defn (free group)

product of words defined similarly

$$\underbrace{(x_{i_1}^{k_1}) \cdots (x_{i_l})^{k_l} (x_{j_1})}^{m_1} \cdots (x_{j_n})^{m_n}$$

Combine $x_i^{a_1} x_i^{a_2} = x_i^{a_1+a_2}$ and use $x_i^0 = e$..

Free group over $\{x_1 \dots x_n\}$. F_n

relations R = subset of F_n

$\langle R \rangle$ minimal normal subgroup containing R.

Then $\langle x_1 \dots x_n / R \rangle$ means F/R.

Lie groups and their discrete subgroups

$$SO(2) = \{ A \in M_2(\mathbb{R}) \mid \langle Ax, Ay \rangle = \langle x, y \rangle, \ \det A = 1 \}$$

$\langle x, y \rangle = x^T y$

for all $x, y \in \mathbb{R}^2$

$$A^T A = I$$

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}, \quad A^T A = I \text{ and } \det A = 1$$

$$\Rightarrow \quad A_\theta = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$

$$A_\theta \cdot A_\varphi = A_{\theta+\varphi}$$

$$\Rightarrow \quad SO(2) \cong \mathbb{R}/\mathbb{Z}$$

$$SO(2) \cong U(1)$$

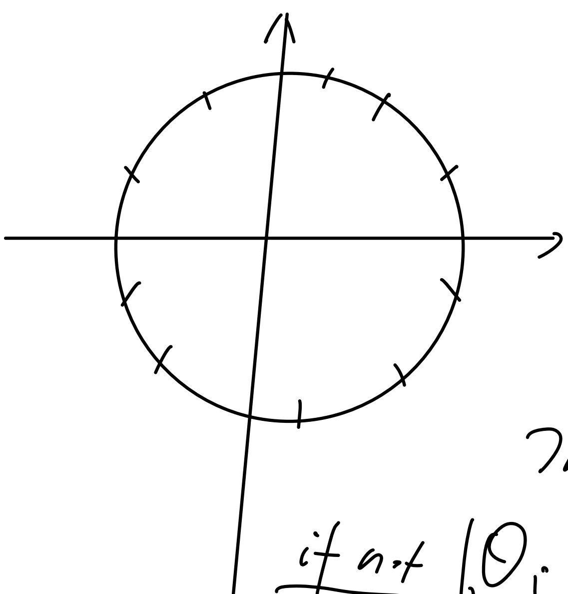
$O(2) = \{ A \in M_2(\mathbb{R}) \mid A^T A = I \}$

$$A_\theta = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \quad \text{or}$$

$$R_\varphi = \begin{bmatrix} \cos\varphi & \sin\varphi \\ \sin\varphi & -\cos\varphi \end{bmatrix}$$

Finite subgroup in $SO(2)$

$G \subset SO(2)$ $\quad G = \{ e, A_{\theta_1}, \dots, A_{\theta_n} \}$



$\theta_i \in [0, 2\pi)$

Find minimal θ_i, say θ_1

Then claim $\theta_i \in \mathbb{Z}\theta_1$

if not $\Rightarrow$ $\theta_i = k\theta_1 + \theta_0$, $\theta_0 \neq 0$.

$\exists k \in \mathbb{Z}$. and $0 < \theta_0 < \theta_1$

$A_{\theta_0} = (A_{\theta_i})(A_{\theta_1})^{-k} \in G$. contradiction.

$\Rightarrow G \cong \mathbb{Z}/d\mathbb{Z}$. $\theta_1 = \frac{2\pi}{d}$.

Finite subgroup in $O(2)$

$G \subset SO(2)$ ✓

$G \not\subset SO(2)$, $\exists R_\varphi$, $\det R_\varphi = -1$

Then up to conjugacy in $O(2)$.

$$R_\varphi = \begin{bmatrix} 1 & \\ & -1 \end{bmatrix}.$$ and

$G \cap SO(2) = \langle A_{\frac{2\pi}{d}} \rangle$.

and Claim $G = \langle A_{\frac{2\pi}{d}}, R_\varphi \rangle$

$= D_d$. dihedral group,