Lie groups and their discrete subgroups

$$SO(2) = \{ A \in M_2(\mathbb{R}) \mid \langle Ax, Ay \rangle = \langle x, y \rangle, \ \det A = 1 \}$$

$\langle x, y \rangle = x^T y$ for all $x, y \in \mathbb{R}^2$

$$A^T A = I$$

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}, \quad A^T A = I \text{ and } \det A = 1$$

$$\Rightarrow \quad A_\theta = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$

$$A_\theta \cdot A_\varphi = A_{\theta + \varphi}$$

$$\Rightarrow \quad SO(2) \cong \mathbb{R}/\mathbb{Z}$$

$$SO(2) \cong U(1)$$

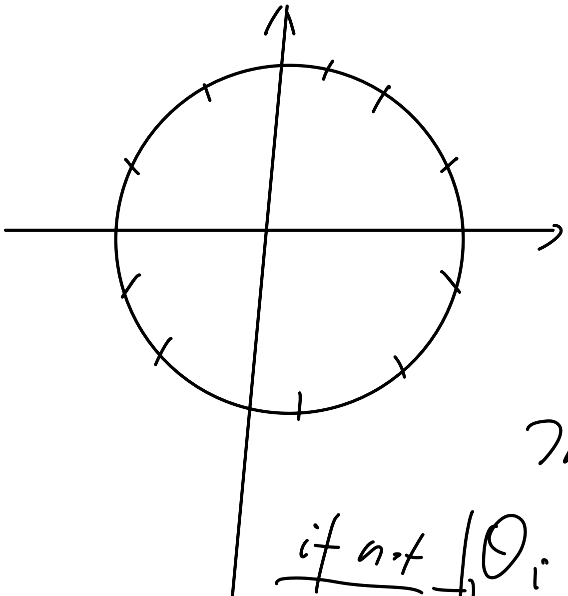
$O(2) = \{ A \in M_2(\mathbb{R}) \mid A^T A = I \}$

$$A_\theta = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \quad \text{or}$$

$$R_\varphi = \begin{bmatrix} \cos\varphi & \sin\varphi \\ \sin\varphi & -\cos\varphi \end{bmatrix}$$

Finite subgroup in $SO(2)$

$G \subset SO(2)$ $\quad G = \{ e, A_{\theta_1}, \dots, A_{\theta_n} \}$



$\theta_i \in [0, 2\pi)$

Find minimal θ_i, say θ_1

Then claim $\theta_i \in \mathbb{Z}\theta_1$

if not $\Rightarrow$ $\theta_i = k\theta_1 + \theta_0$, $\theta_0 \neq 0$.

$\exists k \in \mathbb{Z}$. and $0 < \theta_0 < \theta_1$

$A_{\theta_0} = (A_{\theta_i})(A_{\theta_1})^{-k} \in G$. contradiction.

$\Rightarrow G \cong \mathbb{Z}/d\mathbb{Z}$. $\theta_1 = \frac{2\pi}{d}$.

Finite subgroup in $O(2)$

$G \subset SO(2)$ ✓

$G \not\subset SO(2)$, $\exists R_\varphi$, $\det R_\varphi = -1$

Then up to conjugacy in $O(2)$.

$R_\varphi = \begin{bmatrix} 1 & \\ & -1 \end{bmatrix}$. and

$G \cap SO(2) = \langle A_{\frac{2\pi}{d}} \rangle$.

and Claim $G = \langle A_{\frac{2\pi}{d}}, R_\varphi \rangle$

$= D_d$. dihedral group.

$$\underline{SO(3)} = \{ A \in M_3(\mathbb{R}) \mid A^T A = I_3, \ \det A = 1 \}$$

A has complex eigenvalue $\lambda \in \mathbb{C}$.

$A \cdot v = \lambda v, \quad v \neq 0 \in \mathbb{C}^3$

$$\overline{(A \cdot v)}^T \cdot (Av) = \bar{v}^T \cdot v \neq 0$$

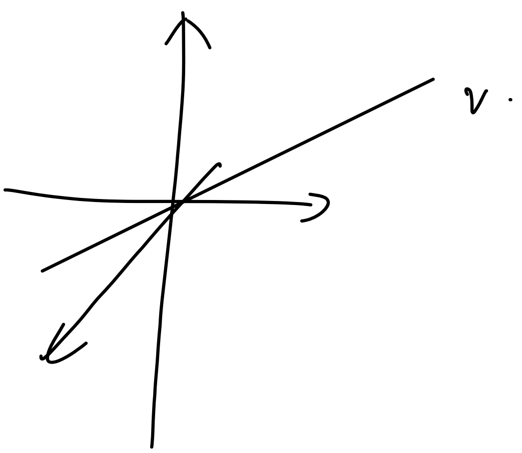
$$\| \qquad\qquad$$

$$\lambda \cdot \bar{\lambda} \, \bar{v}^T v \quad \Rightarrow \quad \lambda \cdot \bar{\lambda} = 1.$$

$|\lambda|^2 = 1.$

If $\lambda \in \mathbb{R}$, $\lambda = \pm 1$, $v \in \mathbb{R}^3$.

$(\mathbb{R} v)^\perp$ is also A-invariant.



and A acts as rotation or reflection depending on $\lambda = 1$, or -1

If $\lambda \notin \mathbb{R}$, $\lambda = \cos\theta + \sqrt{-1}\sin\theta$.

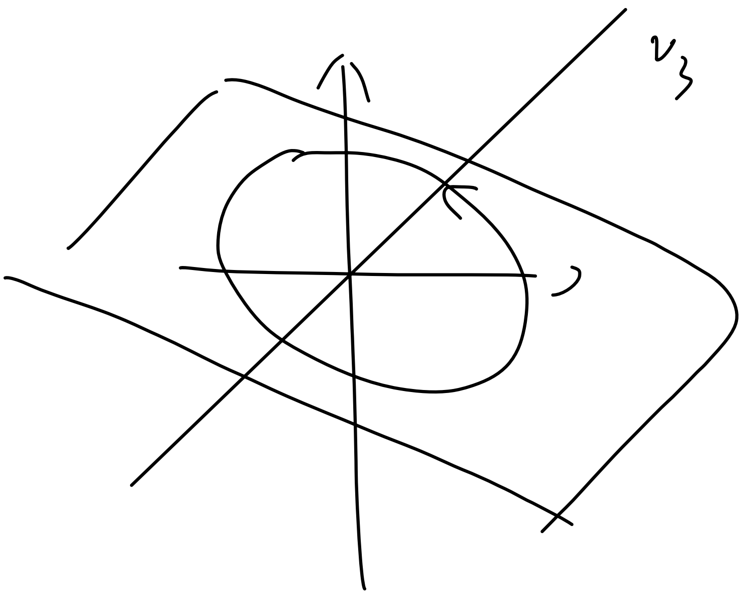
$v = v_1 + \sqrt{-1}\, v_2$, $v_i \in \mathbb{R}^2$

if $v_1 = k v_2$, $\Rightarrow$ $A v_1 = \lambda v_1$ $\Rightarrow$ $\lambda \in \mathbb{R}$.

Then v_1, v_2 $\mathbb{R}$-linearly independent

$$A \cdot (v_1, v_2) = (v_1, v_2) \begin{bmatrix} \cos\theta & +\sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$$

$W = \mathrm{span}(v_1, v_2)$. $\mathbb{R} v_3 = W^\perp$.



To summarize. $A \in SO(3)$ is always a rotation finite subgroups $G \subset SO(3)$ are the following

① C_n.

② D_n

③ T = rotation symmetry group of Tetrahedra

④ O = of cube

⑤ I = of icosahedron

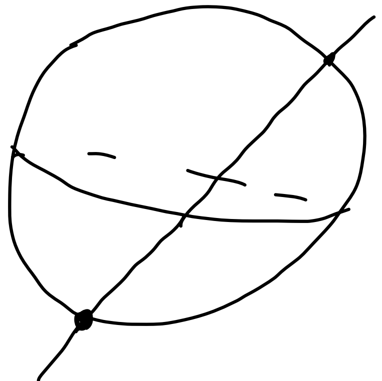
Pf:

$$P^{-1}AP = \begin{bmatrix} \cos\theta & \sin\theta & \\ -\sin\theta & \cos\theta & \\ & & 1 \end{bmatrix} \qquad P = \begin{bmatrix} | & | & | \\ \alpha_1 & \alpha_2 & \alpha_3 \\ | & | & | \end{bmatrix}$$

$\alpha_1, \alpha_2, \alpha_3$ orthonormal

Such $\pm\alpha_3$ are called poles of A.

BAB^{-1} has poles $\pm B\alpha_3$



$$P = \{ \alpha \mid \alpha \text{ pole of } A \in G, A \neq I_3 \}$$

$G \curvearrowright P$.

$$P = orb_1 \cup orb_2 \cup \cdots \cup orb_m.$$

$r_i = |Stab_G(p)| \quad p \in orb_i$.

$$|orb_i| = \frac{|G|}{r_i}.$$

count $\{(g, p) \mid g \neq 1 \in G,\ p \in P,\ g(p) = p\}$

$$2(|G|-1) = \sum_{p \in P} (r_p - 1)$$

$$= \sum_{i=1}^{m} |orb_i|(r_i - 1)$$

$$= \sum_{i=1}^{m} \left(|G| - \frac{|G|}{r_i}\right)$$

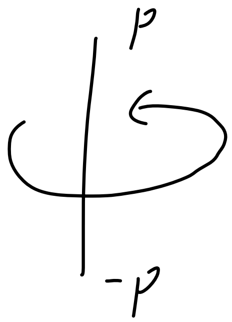
$$2 - \frac{2}{|G|} = \sum_{i=1}^{m} \left(1 - \frac{1}{r_i}\right)$$

$$\frac{1}{r_1} + \cdots + \frac{1}{r_m} = m - 2 + \frac{2}{|G|}.$$

$\underline{r_i \geq 2}$ $\quad m = 1,\ m \geq 4$ 不可能

$m = 2$, $\quad \frac{1}{r_1} + \frac{1}{r_2} = \frac{2}{|G|}$

$r_1 = r_2 = |G|$, $\quad orb_1 = \{p\}$, $\quad orb_2 = \{-p\}$



$G \subset SO(2)$

$m = 3$, $\quad \frac{1}{r_1} + \frac{1}{r_2} + \frac{1}{r_3} = 1 + \frac{2}{|G|} > 1$.

$(r_1, r_2, r_3) = (2, 2, n)$, $\quad D_n$

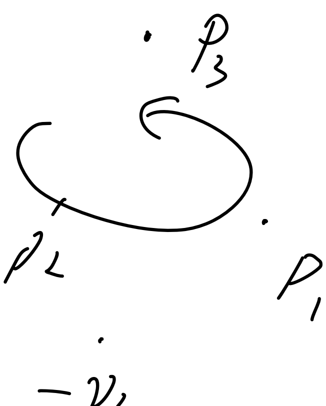
$(2, 3, 3) \quad T$

$(2, 3, 4) \quad O$

$(2, 3, 5) \quad I$.

$(2, 2, n) \longrightarrow \quad |G| = 2n$,

$n \geq 3 \longrightarrow$



$p_1, p_2 \neq \underline{p_3}$, or $\underline{-p_3}$ because $|Stab_{-p_3}| \geq n$

orbit of p_1 is the vertices of a regular n-gon.

$$U(2) = \left\{ A = \begin{pmatrix} a, & b \\ c, & d \end{pmatrix} \in M_2(\mathbb{C}) \;\middle|\; \bar{A}^T \cdot A = I \right\}$$

$$SU(2) = \left\{ A = \begin{pmatrix} a \cdot & b \\ c, & d \end{pmatrix} \in M_2(\mathbb{C}) \;\middle|\; \begin{array}{l} \bar{A}^T A = I \\ \det A = 1 \end{array} \right\}$$

$$A^{-1} = \begin{pmatrix} d & -b \\ -c & a \end{pmatrix} = \bar{A}^T$$

$$\Rightarrow a = \bar{d}, \quad b + \bar{c} = 0$$

$$A = \begin{bmatrix} a & b \\ -\bar{b} & \bar{a} \end{bmatrix}, \quad \underline{\bar{a} a + b \bar{b} = 1}$$